# A First Course in Digital Communications Ha H. Nguyen and E. Shwedyk



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A First Course in Digital Communications

### Introduction

- Though many message sources are inherently digital in nature, two of the most common message sources, audio and video, are analog, i.e., they produce continuous time signals.
- To make analog messages amenable for digital transmission sampling, quantization and encoding are required.
  - *Sampling*: How many samples per second are needed to exactly represent the signal and how to reconstruct the analog message from the samples?
  - *Quantization*: To represent the sample value by a digital symbol chosen from a finite set. What is the choice of a discrete set of amplitudes to represent the continuous range of possible amplitudes and how to measure the distortion due to quantization?
  - *Encoding*: Map the quantized signal sample into a string of digital, typically binary, symbols.

Chapter 4: Sampling and Quantization

Ideal (or Impulse) Sampling

$$m(t)$$

$$m_{s}(t) = \sum_{n=-\infty}^{\infty} m(nT_{s})\delta(t - nT_{s})$$

$$s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_{s})$$
(a)

- *T<sub>s</sub>* is the period of the impulse train, also referred to as the *sampling period*.
- The inverse of the sampling period,  $f_s = 1/T_s$ , is called the sampling frequency or *sampling rate*.
- It is intuitive that the higher the sampling rate is, the more accurate the representation of m(t) by  $m_s(t)$  is.
- What is the minimum sampling rate for the sampled version  $m_s(t)$  to exactly represent the original analog signal m(t)?

### Illustration of Ideal Sampling



### Spectrum of the Sampled Waveform

$$m_s(t) = m(t)s(t) \leftrightarrow M_s(f) = M(f) * S(f)$$

$$M_s(f) = M(f) * \underbrace{\left[\frac{1}{T_s} \sum_{n=-\infty}^{\infty} \delta(f - nf_s)\right]}_{S(f)} = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} M(f - nf_s).$$

- If the bandwidth of m(t) is limited to W Hertz, m(t) can be completely recovered from  $m_s(t)$  by an ideal lowpass filter of bandwidth W if  $f_s \ge 2W$ .
- When f<sub>s</sub> < 2W (under-sampling), the copies of M(f) overlap and it is not possible to recover m(t) by filtering ⇒ aliasing.

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# Reconstruction of m(t)

$$M_{s}(f) = \mathcal{F}\{m_{s}(t)\} = \sum_{n=-\infty}^{\infty} m(nT_{s})\mathcal{F}\{\delta(t-nT_{s})\} = \sum_{n=-\infty}^{\infty} m(nT_{s})\exp(-j2\pi nfT_{s})$$

$$M(f) = \frac{M_{s}(f)}{f_{s}} = \frac{1}{f_{s}}\sum_{n=-\infty}^{\infty} m(nT_{s})\exp(-j2\pi nfT_{s}), \quad -W \le f \le W.$$

$$m(t) = \mathcal{F}^{-1}\{M(f)\} = \int_{-\infty}^{\infty} M(f)\exp(j2\pi ft)df$$

$$= \int_{-W}^{W} \frac{1}{f_{s}}\sum_{n=-\infty}^{\infty} m(nT_{s})\exp(-j2\pi nfT_{s})\exp(j2\pi ft)df$$

$$= \frac{1}{f_{s}}\sum_{n=-\infty}^{\infty} m(nT_{s})\int_{-W}^{W}\exp[j2\pi f(t-nT_{s})]df$$

$$= \sum_{n=-\infty}^{\infty} m(nT_{s})\frac{\sin[2\pi W(t-nT_{s})]}{\pi f_{s}(t-nT_{s})} = \sum_{n=-\infty}^{\infty} m\left(\frac{n}{2W}\right)\operatorname{sinc}(2Wt-n)$$

# Sampling Theorem

#### Theorem

A signal having no frequency components above W Hertz is completely described by specifying the values of the signal at periodic time instants that are separated by at most 1/2Wseconds.

- $f_s \ge 2W$  is known as the Nyquist criterion, the sampling rate  $f_s = 2W$  is called the Nyquist rate and its reciprocal called the Nyquist interval.
- Ideal sampling is not practical ⇒ Need practical sampling methods.

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### **Bandlimited Interpolation**



# Natural Sampling



• In the above, h(t) = 1 for  $0 \le t \le \tau$  and h(t) = 0 otherwise.

- The pulse train p(t) is also known as the gating waveform.
- Natural sampling requires only an on/off gate.

# Illustration of Natural Sampling



### Signal Reconstruction in Natural Sampling

Write the periodic pulse train p(t) in a Fourier series as:

$$p(t) = \sum_{n=-\infty}^{\infty} D_n \exp(j2\pi n f_s t), \quad D_n = \frac{\tau}{T_s} \operatorname{sinc}\left(\frac{n\tau}{T_s}\right) e^{-j\pi n\tau/T_s}.$$

The sampled waveform and its Fourier transform are

$$m_s(t) = m(t) \sum_{n=-\infty}^{\infty} D_n \exp(j2\pi n f_s t).$$

$$M_s(f) = \sum_{n=-\infty}^{\infty} D_n \mathcal{F}\{m(t)\exp(j2\pi nf_s t)\} = \sum_{n=-\infty}^{\infty} D_n M(f - nf_s).$$

The original signal m(t) can still be reconstructed using a lowpass filter as long as the Nyquist criterion is satisfied.

# Flat-Top Sampling

Flat-top sampling is the most popular sampling method and involves two simple operations: *sample* and *hold*.



### Chapter 4: Sampling and Quantization

# Spectrum of $m_s(t)$ in Flat-Top Sampling

# Equalization

- Not possible to reconstruct m(t) using an lowpass filter, even when the Nyquist criterion is satisfied.
- The distortion due to H(f) can be corrected by connecting an *equalizer* in cascade with the lowpass reconstruction filter.
- Ideally, the amplitude response of the equalizer is

$$|H_{\rm eq}| = \frac{T_s}{|H(f)|} = \frac{T_s}{\tau {\rm sinc}(f\tau)}$$



# **Pulse Modulation**

- In pulse modulation, some parameter of a *pulse train* is varied in accordance with the sample values of a message signal.
- Pulse-amplitude modulation (PAM): *amplitudes* of regularly spaced pulses are varied.
  - PAM transmission does not improve the noise performance over baseband modulation, but allows multiplexing, i.e., sharing the same transmission media by different sources.
  - The multiplexing advantage offered by PAM comes at the expense of a larger transmission bandwidth.
- Pulse-width modulation (PWM): *widths* of the individual pulses are varied.
- Pulse-position modulation (PPM): *position* of a pulse relative to its original time of occurrence is varied.
- Pulse modulation techniques are still analog modulation. For digital communications of an analog source, quantization of sampled values is needed.

### PWM & PPM Waveforms with a Sinusoidal Message



# Quantization



- Quantization is to transform  $m(nT_s)$  into a discrete amplitude  $\hat{m}(nT_s)$  taken from a *finite* set.
- If the spacing between two adjacent amplitude levels is sufficiently small, then  $\hat{m}(nT_s)$  can be made practically indistinguishable from  $m(nT_s)$ .
- There is always a loss of information associated with the quantization process, no matter how fine one may choose the finite set of the amplitudes ⇒ Not possible to *completely* recover the sampled signal from the quantized signal.

# Memoryless Quantization



- Quantization of current sample value is independent of earlier/later samples.
- The *l*th interval is determined by the *decision levels* (also called the *threshold levels*)  $D_l$  and  $D_{l+1}$ :

$$\mathcal{I}_l: \{D_l < m \le D_{l+1}\}, \quad l = 1, \dots, L.$$

• Signal amplitudes in  $\mathcal{I}_l$  are all represented by one amplitude  $T_l \in \mathcal{I}_l$  (target level or reconstruction level).

#### Chapter 4: Sampling and Quantization

# Uniform Quantizer

- Step-size is the same and the target level is in the middle of the interval:  $T_l = \frac{D_l + D_{l+1}}{2}$ .
- *Midtread* and *midrise* input/output characteristics:



Chapter 4: Sampling and Quantization

### Input and Output of A Midrise Uniform Quantizer



# Signal-to-Quantization Noise Ratio (SNR<sub>q</sub>)

- Model the input as a zero-mean random variable  ${\bf m}$  with some pdf  $f_{{\bf m}}(m)$ .
- Assume the amplitude range of  $\mathbf{m}$  is  $-m_{\max} \leq \mathbf{m} \leq m_{\max} \Rightarrow$ the quantization step-size is  $\Delta = \frac{2m_{\max}}{L}$ .
- Let  $\mathbf{q} = \mathbf{m} \hat{\mathbf{m}}$  be the quantization error, then  $-\Delta/2 \leq \mathbf{q} \leq \Delta/2$ .
- If  $\Delta$  is sufficiently small (*L* is sufficiently large), **q** is approximately *uniform* over  $[-\Delta/2, \Delta/2]$ :

$$f_{\mathbf{q}}(q) = \left\{ \begin{array}{ll} \frac{1}{\Delta}, & -\frac{\Delta}{2} < q \leq \frac{\Delta}{2} \\ 0, & \text{otherwise} \end{array} \right.$$

• The mean of  ${f q}$  is zero, while its variance is:

$$\sigma_{\mathbf{q}}^2 = \int_{-\Delta/2}^{\Delta/2} q^2 f_{\mathbf{q}}(q) \mathrm{d}q = \int_{-\Delta/2}^{\Delta/2} q^2 \left(\frac{1}{\Delta}\right) \mathrm{d}q = \frac{\Delta^2}{12} = \frac{m_{\max}^2}{3L^2}.$$

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- With  $L = 2^R$ , where R is the number of bits needed to represent each target level, then  $\sigma_{\mathbf{q}}^2 = \frac{m_{\text{max}}^2}{3 \times 2^{2R}}$
- The average message power is  $\sigma_{\mathbf{m}}^2 = \int_{-m_{\text{max}}}^{m_{\text{max}}} m^2 f_{\mathbf{m}}(m) \mathrm{d}m.$
- The signal-to-quantization noise ratio is

$$SNR_{q} = \left(\frac{3\sigma_{\mathbf{m}}^{2}}{m_{\max}^{2}}\right)2^{2R} = \frac{3 \times 2^{2R}}{F^{2}}.$$

• F is called the crest factor of the message, defined as,

$$F = \frac{\text{Peak value of the signal}}{\text{RMS value of the signal}} = \frac{m_{\text{max}}}{\sigma_{\text{m}}}.$$

• SNR<sub>q</sub> increases *exponentially* with the number of bits per sample *R* and decreases with the square of the message's crest factor.

 $\bullet~\ensuremath{\mathsf{Expressed}}$  in decibels,  $\ensuremath{\mathrm{SNR}_q}$  is

$$10 \log_{10} \text{SNR}_{q} = 6.02R + 10 \log_{10} \left(\frac{\sigma_{m}^{2}}{m_{\max}^{2}}\right) + 4.77$$
$$= 6.02R - 20 \log_{10} F + 4.77$$

An additional 6-dB improvement in  $\rm SNR_q$  is obtained for each bit added to represent the continuous signal sample (6-dB rule).

# **Optimal Quantizer**

- Uniform quantizer is not optimal in terms of minimizing the signal-to-quantization noise ratio.
- In general, the decision levels are constrained to satisfy:

$$D_{1} = -m_{\max}, \\ D_{L+1} = m_{\max}, \\ D_{l} \le D_{l+1}, \text{ for } l = 1, 2, \dots L.$$

• The average quantization noise power is

$$N_{q} = \sum_{l=1}^{L} \int_{D_{l}}^{D_{l+1}} (m - T_{l})^{2} f_{\mathbf{m}}(m) dm.$$

• To obtain the optimal quantizer that maximizes the SNR<sub>q</sub>, one needs to find the set of 2L - 1 variables  $\{D_2, D_3, \ldots, D_L, T_1, T_2, \ldots, T_L\}$  to minimize  $N_q$ .

• Differentiate  $N_{\rm q}$  with respect to  $D_j$  and set the result to 0:

$$\frac{\partial N_{\mathbf{q}}}{\partial D_{j}} = f_{\mathbf{m}}(D_{j}) \left[ (D_{j} - T_{j-1})^{2} - (D_{j} - T_{j})^{2} \right] = 0, \ j = 2, 3, \dots L.$$

$$D_l^{\text{opt}} = \frac{T_{l-1} + T_l}{2}, \quad l = 2, 3, \dots L.$$

⇒ The decision levels are the midpoints of the target values! • Differentiate  $N_q$  with respect to  $T_j$  and set the result to 0:

$$\frac{\partial N_{\mathbf{q}}}{\partial T_{j}} = -2 \int_{D_{j}}^{D_{j+1}} (m - T_{j}) f_{\mathbf{m}}(m) \mathrm{d}m = 0, \ j = 1, 2, \dots L.$$

$$T_l^{\text{opt}} = \frac{\int_{D_l}^{D_{l+1}} m f_{\mathbf{m}}(m) \mathrm{d}m}{\int_{D_l}^{D_{l+1}} f_{\mathbf{m}}(m) \mathrm{d}m}, \quad l = 1, 2, \dots, L.$$

 $\Rightarrow$  The target value for a quantization region should be chosen to be the *centroid* of that region.

Chapter 4: Sampling and Quantization

### Example of Optimal Quantizer Design (Problem 4.6)



### Lloyd-Max Conditions and Iterative Algorithm

$$D_l^{\text{opt}} = \frac{T_{l-1} + T_l}{2}, \quad (5) \qquad T_l^{\text{opt}} = \frac{\int_{D_l}^{D_{l+1}} m f_{\mathbf{m}}(m) dm}{\int_{D_l}^{D_{l+1}} f_{\mathbf{m}}(m) dm}. \quad (6)$$

$$l=2,3,\ldots L$$

- Start by specifying an arbitrary set of decision levels (for example the set that results in equal-length regions) and then find the target values using (6).
- Obtermine the new decision levels using (5).
- The two steps are iterated until the parameters do not change significantly from one step to the next.

The optimal quantizer needs to know pdf  $f_{\mathbf{m}}(m)$  and is designed for a specific  $m_{\max} \Rightarrow$  Prefer quantization methods that are robust to source statistics and changes in the signal's power level.

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### **Robust Quantizers**

- When the message signal is uniformly distributed, the optimal quantizer is a uniform quantizer ⇒ As long as the distribution of the message signal is close to uniform, the uniform quantizer works fine.
- For a voice signal, there exists a higher probability for smaller amplitudes and a lower probability for larger amplitudes ⇒ it is more efficient to design a quantizer with more quantization regions at lower amplitudes and less quantization regions at larger amplitudes (i.e., nonuniform quantization).
- Robust method for performing nonuniform quantization is to use *compander=compressor+ expander*.



Chapter 4: Sampling and Quantization

 $\mu$ -law and A-law Companders



Chapter 4: Sampling and Quantization

### $\mathrm{SNR}_{q}$ of Non-Uniform Quantizers



When  $L \gg 1$ ,  $\Delta$  and  $\Delta_l$  are small  $\Rightarrow f_{\mathbf{m}}(m)$  is a constant  $f_{\mathbf{m}}(m_l)$  over  $\Delta_l$  and  $m_l$  is at the midpoint of the *l*th quantization region.

$$N_{\mathbf{q}} = \sum_{l=1}^{L} \int_{m_{l} - \frac{\Delta_{l}}{2}}^{m_{l} + \frac{\Delta_{l}}{2}} (m - m_{l})^{2} f_{\mathbf{m}}(m) dm$$
$$\cong \sum_{l=1}^{L} f_{\mathbf{m}}(m_{l}) \int_{m_{l} - \frac{\Delta_{l}}{2}}^{m_{l} + \frac{\Delta_{l}}{2}} (m - m_{l})^{2} dm = \sum_{l=1}^{L} \frac{\Delta_{l}^{3}}{12} f_{\mathbf{m}}(m_{l}).$$

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$$\frac{\Delta}{\Delta_l} = \frac{\mathrm{d}g(m)}{\mathrm{d}m}\Big|_{m=m_l} \Rightarrow N_{\mathrm{q}} = \frac{\Delta^2}{12} \sum_{l=1}^{L} \frac{f_{\mathrm{m}}(m_l)}{\left(\frac{\mathrm{d}g(m)}{\mathrm{d}m}\Big|_{m=m_l}\right)^2} \Delta_l.$$

Since  $L\gg1,$  approximate the summation by an integral to obtain

$$N_{\rm q} = \frac{\Delta^2}{12} \int_{-m_{\rm max}}^{m_{\rm max}} \frac{f_{\rm m}(m)}{\left(\frac{\mathrm{d}g(m)}{\mathrm{d}m}\right)^2} \mathrm{d}m = \frac{y_{\rm max}^2}{3L^2} \int_{-m_{\rm max}}^{m_{\rm max}} \frac{f_{\rm m}(m)}{\left(\frac{\mathrm{d}g(m)}{\mathrm{d}m}\right)^2} \mathrm{d}m.$$

# Chapter 4: Sampling and Quantization

 $\mathrm{SNR}_{\mathrm{q}}$  of  $\mu$ -law Compander

$$\frac{\mathrm{d}g(m)}{\mathrm{d}m} = \frac{y_{\max}}{\ln(1+\mu)} \frac{\mu(1/m_{\max})}{1+\mu(|m|/m_{\max})}.$$

$$N_{\mathrm{q}} = \frac{y_{\max}^2}{3L^2} \frac{\ln^2(1+\mu)}{y_{\max}^2 \left(\frac{\mu}{m_{\max}}\right)^2} \int_{-m_{\max}}^{m_{\max}} \left[1+\mu\left(\frac{|m|}{m_{\max}}\right)\right]^2 f_{\mathrm{m}}(m) \mathrm{d}m$$

$$\frac{m_{\max}^2}{3L^2} \frac{\ln^2(1+\mu)}{\mu^2} \int_{-m_{\max}}^{m_{\max}} \left[1+2\mu\left(\frac{|m|}{m_{\max}}\right)+\mu^2\left(\frac{|m|}{m_{\max}}\right)^2\right] f_{\mathrm{m}}(m) \mathrm{d}m.$$
Since  $\int_{-m_{\max}}^{m_{\max}} f_{\mathrm{m}}(m) \mathrm{d}m = 1, \int_{-m_{\max}}^{m_{\max}} m^2 f_{\mathrm{m}}(m) \mathrm{d}m = \sigma_{\mathrm{m}}^2$  and

Since 
$$\int_{-m_{\max}} \int_{\mathbf{m}}^{m_{\max}} \int_{\mathbf{m}}^{m_{\max}} m f_{\mathbf{m}}(m) dm = 0$$
 m and  $\int_{-m_{\max}}^{m_{\max}} |m| f_{\mathbf{m}}(m) dm = E\{|\mathbf{m}|\}, \text{ then}$ 

$$N_{\rm q} = \frac{m_{\rm max}^2}{3L^2} \frac{\ln^2(1+\mu)}{\mu^2} \left[ 1 + 2\mu \frac{E\{|\mathbf{m}|\}}{m_{\rm max}} + \mu^2 \frac{\sigma_{\mathbf{m}}^2}{m_{\rm max}^2} \right].$$

$$SNR_{q} = \frac{\sigma_{\mathbf{m}}^{2}}{N_{q}} = \frac{3L^{2}\mu^{2}}{\ln^{2}(1+\mu)} \frac{(\sigma_{\mathbf{m}}^{2}/m_{\max}^{2})}{1+2\mu(E\{|\mathbf{m}|\}/m_{\max})+\mu^{2}(\sigma_{\mathbf{m}}^{2}/m_{\max}^{2})}.$$

Define 
$$\sigma_n^2 = \frac{\sigma_{\mathbf{m}}^2}{m_{\max}^2}$$
, then  $\frac{E\{|\mathbf{m}|\}}{\sigma_{\mathbf{m}}} \frac{\sigma_{\mathbf{m}}}{m_{\max}} = \frac{E\{|\mathbf{m}|\}}{\sigma_{\mathbf{m}}} \sigma_n$ . Therefore,  
 $\operatorname{SNR}_{\mathbf{q}}(\sigma_n^2) = \frac{3L^2\mu^2}{\ln^2(1+\mu)} \frac{\sigma_n^2}{1+2\mu\sigma_n \frac{E\{|\mathbf{m}|\}}{\sigma_{\mathbf{m}}} + \mu^2\sigma_n^2}.$ 

If  $\mu \gg 1$  then the dependence of  $SNR_q$  on the message's characteristics is very small and  $SNR_q$  can be approximated as

$$\mathrm{SNR}_{\mathrm{q}} = \frac{3L^2}{\ln^2(1+\mu)}.$$

For practical values of  $\mu = 255$  and L = 256,  $SNR_q = 38.1 dB$ .

# 8-bit Quantizer for the Gaussian-Distributed Message



One sacrifices performance for larger input power levels to obtain a performance that remains robust over a wide range of input levels.

# ${ m SNR}_{ m q}$ with 8-bit $\mu$ -law quantizer (L=256, $\mu=255$ )



Insensitive to variations in input signal power and also insensitive to the actual pdf model – Both desirable properties.

# **Differential Quantizers**

- Most message signals (e.g., voice or video) exhibit a high degree of correlation between successive samples.
- Redundancy can be exploited to obtain a better SNR<sub>q</sub> for a given *L*, or conversely for a specified SNR<sub>q</sub> the number of levels *L* can be reduced:
  - Use the previous sample values to predict the next sample value and then transmit the difference.
  - Quantize and transmit the prediction error,

$$\mathbf{e}[n] = \mathbf{m}[n] - \tilde{\mathbf{m}}[n].$$



If  $|e_{\max}| = |m_{\max} - km_{\max}| = |1 - k|m_{\max}$  is less than  $m_{\max}$  then the quantization noise power is reduced!

### Linear Predictor



Select  $\{w_i\}$  to *minimize* the variance of prediction error:

$$\sigma_{\mathbf{e}}^{2} = E\left\{\left(\mathbf{m}[n] - \sum_{i=1}^{p} w_{i}\mathbf{m}[n-i]\right)^{2}\right\} = E\{\mathbf{m}^{2}[n]\} - 2\sum_{i=1}^{p} w_{i}E\{\mathbf{m}[n]\mathbf{m}[n-i]\} + \sum_{i=1}^{p} \sum_{j=1}^{p} w_{i}w_{j}E\{\mathbf{m}[n-i]\mathbf{m}[n-j]\}.$$

### Normal Equations (or the Yule-Walker Equations)

• With  $R_{\mathbf{m}}(k) = E\{\mathbf{m}[n]\mathbf{m}[n+k]\}$  the *autocorrelation* of  $\{\mathbf{m}[n]\}$ ,

$$\sigma_{\mathbf{e}}^2 = R_{\mathbf{m}}(0) - 2\sum_{i=1}^p w_i R_{\mathbf{m}}(i) + \sum_{i=1}^p \sum_{j=1}^p w_i w_j R_{\mathbf{m}}(i-j).$$

 Take the partial derivative of σ<sup>2</sup><sub>e</sub> with respect to each coefficient w<sub>i</sub> and set the results to zero to yield:

$$\begin{bmatrix} R_{\mathbf{m}}(0) & R_{\mathbf{m}}(1) & R_{\mathbf{m}}(2) & \cdots & R_{\mathbf{m}}(p-1) \\ R_{\mathbf{m}}(-1) & R_{\mathbf{m}}(0) & R_{\mathbf{m}}(1) & \cdots & R_{\mathbf{m}}(p-2) \\ R_{\mathbf{m}}(-2) & R_{\mathbf{m}}(-1) & R_{\mathbf{m}}(0) & \cdots & R_{\mathbf{m}}(p-3) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ R_{\mathbf{m}}(-p+1) & R_{\mathbf{m}}(-p+2) & R_{\mathbf{m}}(-p+3) & \cdots & R_{\mathbf{m}}(0) \end{bmatrix}.$$

$$\begin{bmatrix} w_{1} \\ w_{2} \\ w_{3} \\ \vdots \\ w_{p} \end{bmatrix} = \begin{bmatrix} R_{\mathbf{m}}(1) \\ R_{\mathbf{m}}(2) \\ R_{\mathbf{m}}(3) \\ \vdots \\ R_{\mathbf{m}}(p) \end{bmatrix}.$$

### Reconstruction of m[n] from the Differential Samples

Ignore the quantization error and look at the reconstruction of  $\mathbf{m}[n]$  from the differential samples  $\mathbf{e}[n]$ .

$$\mathbf{e}[n] = \mathbf{m}[n] - \sum_{i=1}^{p} w_i \mathbf{m}[n-i].$$

$$\mathbf{e}(z^{-1}) = \mathbf{m}(z^{-1}) - \sum_{i=1}^{p} w_i z^{-i} \mathbf{m}(z^{-1}) = \mathbf{m}(z^{-1}) - \mathbf{m}(z^{-1}) \sum_{i=1}^{p} w_i z^{-i}$$
$$= \mathbf{m}(z^{-1}) - \mathbf{m}(z^{-1}) H(z^{-1}) \Rightarrow \mathbf{m}(z^{-1}) = \frac{1}{1 - H(z^{-1})} \mathbf{e}(z^{-1}).$$



Under quantization noise error, use DPCM to eliminate the effect of previous quantization noise samples.



$$\hat{\mathbf{m}}[n] = \widetilde{\mathbf{m}}[n] + \hat{\mathbf{e}}[n] = \widetilde{\mathbf{m}}[n] + (\mathbf{e}[n] - \mathbf{q}[n]) = (\widetilde{\mathbf{m}}[n] + \mathbf{e}[n]) - \mathbf{q}[n] = \mathbf{m}[n] - \mathbf{q}[n].$$

# Pulse-Code Modulation (PCM)

- A PCM signal is obtained from the quantized PAM signal by encoding each quantized sample to a *digital codeword*.
- In binary PCM each quantized sample is digitally encoded into an *R*-bit binary codeword, where  $R = \lceil \log_2 L \rceil + 1$ .
- Binary digits of a PCM signal can be transmitted using many efficient modulation schemes.
- There are several mappings: Natural binary coding (NBC), *Gray* mapping, foldover binary coding (FBC), etc.

