

A First Course in Digital Communications

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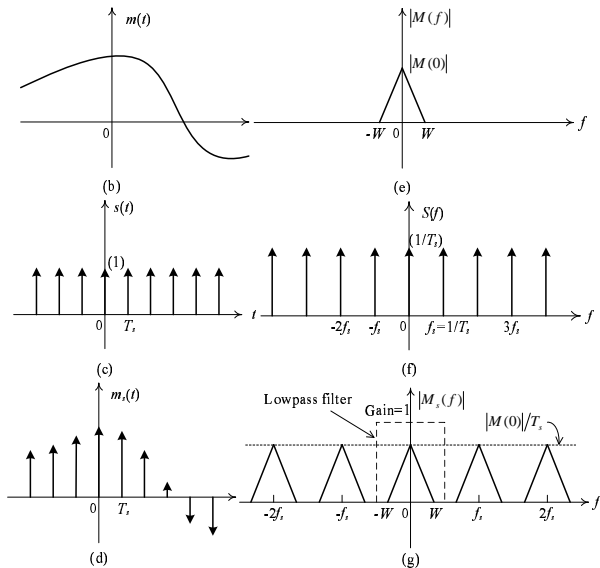
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Introduction

- Though many message sources are inherently digital in nature, two of the most common message sources, audio and video, are analog, i.e., they produce continuous time signals.
- To make analog messages amenable for digital transmission sampling, quantization and encoding are required.
 - *Sampling*: How many samples per second are needed to exactly represent the signal and how to reconstruct the analog message from the samples?
 - *Quantization*: To represent the sample value by a digital symbol chosen from a finite set. What is the choice of a discrete set of amplitudes to represent the continuous range of possible amplitudes and how to measure the distortion due to quantization?
 - *Encoding*: Map the quantized signal sample into a string of digital, typically binary, symbols.

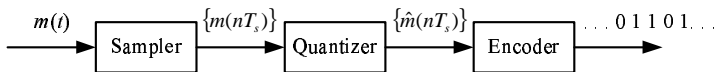
Illustration of Ideal Sampling



Pulse Modulation

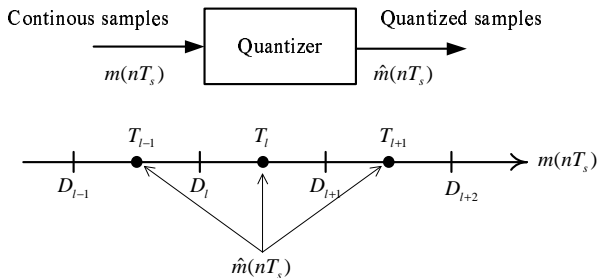
- In pulse modulation, some parameter of a *pulse train* is varied in accordance with the sample values of a message signal.
- Pulse-amplitude modulation (PAM): *amplitudes* of regularly spaced pulses are varied.
 - PAM transmission does not improve the noise performance over baseband modulation, but allows multiplexing, i.e., sharing the same transmission media by different sources.
 - The multiplexing advantage offered by PAM comes at the expense of a larger transmission bandwidth.
- Pulse-width modulation (PWM): *widths* of the individual pulses are varied.
- Pulse-position modulation (PPM): *position* of a pulse relative to its original time of occurrence is varied.
- Pulse modulation techniques are still analog modulation. For digital communications of an analog source, quantization of sampled values is needed.

Quantization



- Quantization is to transform $m(nT_s)$ into a discrete amplitude $\hat{m}(nT_s)$ taken from a *finite* set.
- If the spacing between two adjacent amplitude levels is sufficiently small, then $\hat{m}(nT_s)$ can be made practically indistinguishable from $m(nT_s)$.
- There is always a loss of information associated with the quantization process, no matter how fine one may choose the finite set of the amplitudes \Rightarrow Not possible to *completely* recover the sampled signal from the quantized signal.

Memoryless Quantization



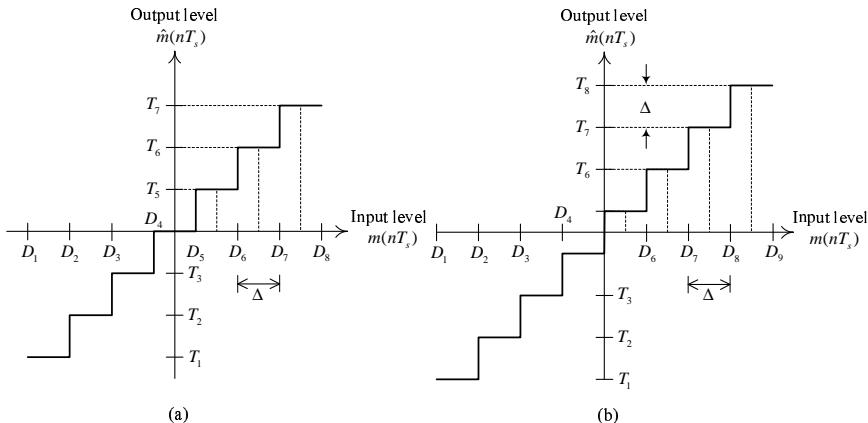
- Quantization of current sample value is independent of earlier/later samples.
- The l th interval is determined by the *decision levels* (also called the *threshold levels*) D_l and D_{l+1} :

$$\mathcal{I}_l : \{D_l < m \leq D_{l+1}\}, \quad l = 1, \dots, L.$$

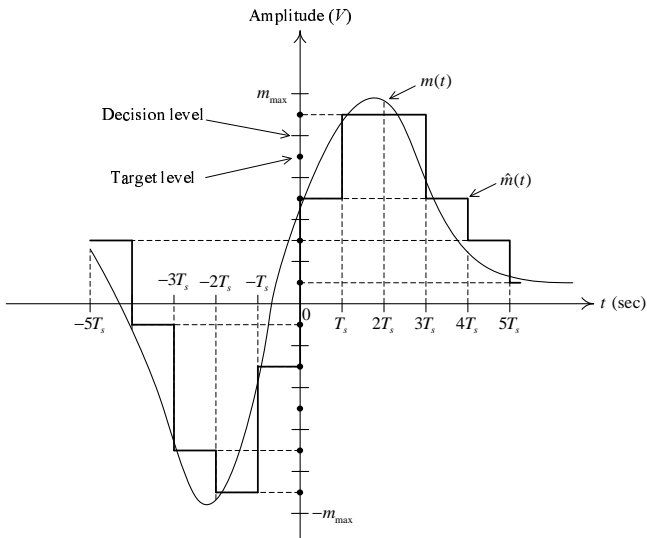
- Signal amplitudes in \mathcal{I}_l are all represented by one amplitude $T_l \in \mathcal{I}_l$ (*target level* or *reconstruction level*).

Uniform Quantizer

- *Step-size* is the same and the target level is in the middle of the interval: $T_l = \frac{D_l + D_{l+1}}{2}$.
- *Midtread* and *midrise* input/output characteristics:



Input and Output of A Midrise Uniform Quantizer



Signal-to-Quantization Noise Ratio (SNR_q)

- Model the input as a zero-mean random variable \mathbf{m} with some pdf $f_{\mathbf{m}}(m)$.
- Assume the amplitude range of \mathbf{m} is $-m_{\max} \leq \mathbf{m} \leq m_{\max} \Rightarrow$ the quantization step-size is $\Delta = \frac{2m_{\max}}{L}$.
- Let $\mathbf{q} = \mathbf{m} - \hat{\mathbf{m}}$ be the quantization error, then $-\Delta/2 \leq \mathbf{q} \leq \Delta/2$.
- If Δ is sufficiently small (L is sufficiently large), \mathbf{q} is approximately *uniform* over $[-\Delta/2, \Delta/2]$:

$$f_{\mathbf{q}}(q) = \begin{cases} \frac{1}{\Delta}, & -\frac{\Delta}{2} < q \leq \frac{\Delta}{2} \\ 0, & \text{otherwise} \end{cases}.$$

- The mean of \mathbf{q} is zero, while its variance is:

$$\sigma_{\mathbf{q}}^2 = \int_{-\Delta/2}^{\Delta/2} q^2 f_{\mathbf{q}}(q) dq = \int_{-\Delta/2}^{\Delta/2} q^2 \left(\frac{1}{\Delta} \right) dq = \frac{\Delta^2}{12} = \frac{m_{\max}^2}{3L^2}.$$

- With $L = 2^R$, where R is the number of bits needed to represent each target level, then $\sigma_{\mathbf{q}}^2 = \frac{m_{\max}^2}{3 \times 2^{2R}}$
- The average message power is $\sigma_{\mathbf{m}}^2 = \int_{-m_{\max}}^{m_{\max}} m^2 f_{\mathbf{m}}(m) dm$.
- The signal-to-quantization noise ratio is

$$\text{SNR}_{\mathbf{q}} = \left(\frac{3\sigma_{\mathbf{m}}^2}{m_{\max}^2} \right) 2^{2R} = \frac{3 \times 2^{2R}}{F^2}.$$

- F is called the *crest factor* of the message, defined as,

$$F = \frac{\text{Peak value of the signal}}{\text{RMS value of the signal}} = \frac{m_{\max}}{\sigma_{\mathbf{m}}}.$$

- $\text{SNR}_{\mathbf{q}}$ increases *exponentially* with the number of bits per sample R and decreases with the square of the message's crest factor.

- Expressed in decibels, SNR_q is

$$\begin{aligned} 10 \log_{10} \text{SNR}_q &= 6.02R + 10 \log_{10} \left(\frac{\sigma_m^2}{m_{\max}^2} \right) + 4.77 \\ &= 6.02R - 20 \log_{10} F + 4.77 \end{aligned}$$

An additional 6-dB improvement in SNR_q is obtained for each bit added to represent the continuous signal sample (6-dB rule).

Optimal Quantizer

- Uniform quantizer is not optimal in terms of minimizing the signal-to-quantization noise ratio.
- In general, the decision levels are constrained to satisfy:

$$\begin{aligned} D_1 &= -m_{\max}, \\ D_{L+1} &= m_{\max}, \\ D_l &\leq D_{l+1}, \quad \text{for } l = 1, 2, \dots, L. \end{aligned}$$

- The average quantization noise power is

$$N_q = \sum_{l=1}^L \int_{D_l}^{D_{l+1}} (m - T_l)^2 f_{\mathbf{m}}(m) dm.$$

- To obtain the optimal quantizer that maximizes the SNR_q , one needs to find the set of $2L - 1$ variables $\{D_2, D_3, \dots, D_L, T_1, T_2, \dots, T_L\}$ to minimize N_q .

- Differentiate N_q with respect to D_j and set the result to 0:

$$\frac{\partial N_q}{\partial D_j} = f_{\mathbf{m}}(D_j) [(D_j - T_{j-1})^2 - (D_j - T_j)^2] = 0, \quad j = 2, 3, \dots, L.$$

$$D_l^{\text{opt}} = \frac{T_{l-1} + T_l}{2}, \quad l = 2, 3, \dots, L.$$

⇒ The decision levels are the midpoints of the target values!

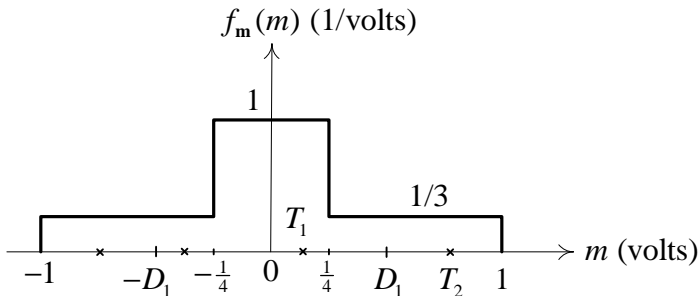
- Differentiate N_q with respect to T_j and set the result to 0:

$$\frac{\partial N_q}{\partial T_j} = -2 \int_{D_j}^{D_{j+1}} (m - T_j) f_{\mathbf{m}}(m) dm = 0, \quad j = 1, 2, \dots, L.$$

$$T_l^{\text{opt}} = \frac{\int_{D_l}^{D_{l+1}} m f_{\mathbf{m}}(m) dm}{\int_{D_l}^{D_{l+1}} f_{\mathbf{m}}(m) dm}, \quad l = 1, 2, \dots, L.$$

⇒ The target value for a quantization region should be chosen to be the *centroid* of that region.

Example of Optimal Quantizer Design (Problem 4.6)



$$T_1 = \frac{\int_0^{D_1} m f_{\mathbf{m}}(m) dm}{\int_0^{D_1} f_{\mathbf{m}}(m) dm} = \frac{\int_0^{1/4} m dm + \frac{1}{3} \int_{1/4}^{D_1} m dm}{\frac{1}{4} + (D_1 - \frac{1}{4}) \frac{1}{3}} = \frac{1 + 8D_1^2}{8 + 16D_1} \quad (1)$$

$$T_2 = \frac{\int_{D_1}^1 m f_{\mathbf{m}}(m) dm}{\int_{D_1}^1 f_{\mathbf{m}}(m) dm} = \frac{1 - D_1^2}{2(1 - D_1)} = \frac{1 + D_1}{2}, \quad D_1 = \frac{T_1 + T_2}{2} \quad (2)$$

$$\therefore 2D_1 = \frac{1 + 8D_1^2}{8 + 16D_1} + \frac{1 + D_1}{2} \Rightarrow 4D_1^2 - D_1 + \frac{5}{4} = 0 \Rightarrow D_1 = 0.4478 \quad (3)$$

$$T_1 = 0.1717; \quad T_2 = 0.7239. \quad (4)$$

Lloyd-Max Conditions and Iterative Algorithm

$$D_l^{\text{opt}} = \frac{T_{l-1} + T_l}{2}, \quad (5) \quad T_l^{\text{opt}} = \frac{\int_{D_l}^{D_{l+1}} m f_{\mathbf{m}}(m) dm}{\int_{D_l}^{D_{l+1}} f_{\mathbf{m}}(m) dm}. \quad (6)$$

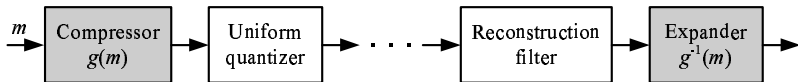
$$l = 2, 3, \dots, L$$

- 1 Start by specifying an arbitrary set of decision levels (for example the set that results in equal-length regions) and then find the target values using (6).
- 2 Determine the new decision levels using (5).
- 3 The two steps are iterated until the parameters do not change significantly from one step to the next.

The optimal quantizer needs to know pdf $f_{\mathbf{m}}(m)$ and is designed for a specific m_{max} \Rightarrow Prefer quantization methods that are robust to source statistics and changes in the signal's power level.

Robust Quantizers

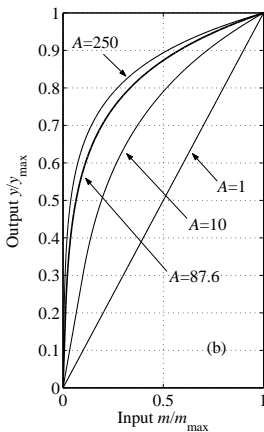
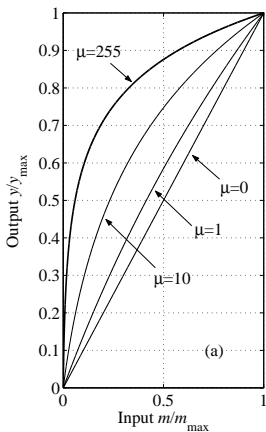
- When the message signal is uniformly distributed, the optimal quantizer is a uniform quantizer \Rightarrow As long as the distribution of the message signal is close to uniform, the uniform quantizer works fine.
- For a voice signal, there exists a higher probability for smaller amplitudes and a lower probability for larger amplitudes \Rightarrow it is more efficient to design a quantizer with more quantization regions at lower amplitudes and less quantization regions at larger amplitudes (i.e., nonuniform quantization).
- Robust method for performing nonuniform quantization is to use *compander* = *compressor* + *expander*.



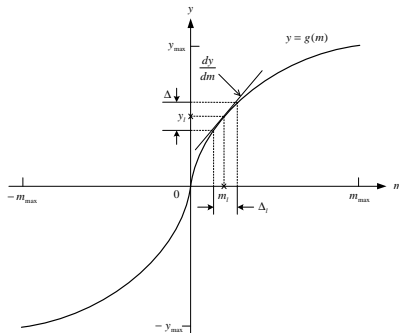
μ -law and A -law Companders

$$y = y_{\max} \frac{\ln[1 + \mu(|m|/m_{\max})]}{\ln(1 + \mu)} \operatorname{sgn}(m), \quad (\mu\text{-law})$$

$$y = \begin{cases} y_{\max} \frac{A(|m|/m_{\max})}{1 + \ln A} \operatorname{sgn}(m), & 0 < \frac{|m|}{m_{\max}} \leq \frac{1}{A} \\ y_{\max} \frac{1 + \ln[A(|m|/m_{\max})]}{1 + \ln A} \operatorname{sgn}(m), & \frac{1}{A} < \frac{|m|}{m_{\max}} < 1 \end{cases}, \quad (A\text{-law})$$

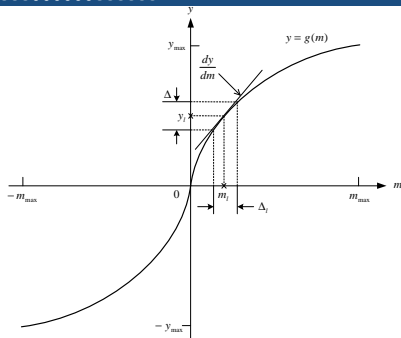


SNR_q of Non-Uniform Quantizers



When $L \gg 1$, Δ and Δ_l are small $\Rightarrow f_{\mathbf{m}}(m)$ is a constant $f_{\mathbf{m}}(m_l)$ over Δ_l and m_l is at the midpoint of the l th quantization region.

$$\begin{aligned}
 N_q &= \sum_{l=1}^L \int_{m_l - \frac{\Delta_l}{2}}^{m_l + \frac{\Delta_l}{2}} (m - m_l)^2 f_{\mathbf{m}}(m) dm \\
 &\cong \sum_{l=1}^L f_{\mathbf{m}}(m_l) \int_{m_l - \frac{\Delta_l}{2}}^{m_l + \frac{\Delta_l}{2}} (m - m_l)^2 dm = \sum_{l=1}^L \frac{\Delta_l^3}{12} f_{\mathbf{m}}(m_l).
 \end{aligned}$$



$$\frac{\Delta}{\Delta_l} = \left. \frac{dg(m)}{dm} \right|_{m=m_l} \Rightarrow N_q = \frac{\Delta^2}{12} \sum_{l=1}^L \frac{f_{\mathbf{m}}(m_l)}{\left(\left. \frac{dg(m)}{dm} \right|_{m=m_l} \right)^2} \Delta_l.$$

Since $L \gg 1$, approximate the summation by an integral to obtain

$$N_q = \frac{\Delta^2}{12} \int_{-m_{\max}}^{m_{\max}} \frac{f_{\mathbf{m}}(m)}{\left(\frac{dg(m)}{dm} \right)^2} dm = \frac{y_{\max}^2}{3L^2} \int_{-m_{\max}}^{m_{\max}} \frac{f_{\mathbf{m}}(m)}{\left(\frac{dg(m)}{dm} \right)^2} dm.$$

SNR_q of μ -law Compander

$$\frac{dg(m)}{dm} = \frac{y_{\max}}{\ln(1 + \mu)} \frac{\mu(1/m_{\max})}{1 + \mu(|m|/m_{\max})}.$$

$$N_q = \frac{y_{\max}^2}{3L^2} \frac{\ln^2(1 + \mu)}{y_{\max}^2 \left(\frac{\mu}{m_{\max}}\right)^2} \int_{-m_{\max}}^{m_{\max}} \left[1 + \mu \left(\frac{|m|}{m_{\max}} \right) \right]^2 f_{\mathbf{m}}(m) dm$$

$$= \frac{m_{\max}^2}{3L^2} \frac{\ln^2(1 + \mu)}{\mu^2} \int_{-m_{\max}}^{m_{\max}} \left[1 + 2\mu \left(\frac{|m|}{m_{\max}} \right) + \mu^2 \left(\frac{|m|}{m_{\max}} \right)^2 \right] f_{\mathbf{m}}(m) dm.$$

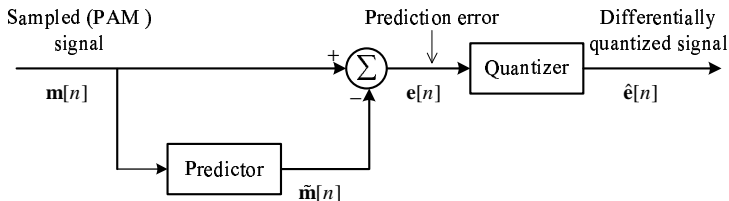
Since $\int_{-m_{\max}}^{m_{\max}} f_{\mathbf{m}}(m) dm = 1$, $\int_{-m_{\max}}^{m_{\max}} m^2 f_{\mathbf{m}}(m) dm = \sigma_{\mathbf{m}}^2$ and $\int_{-m_{\max}}^{m_{\max}} |m| f_{\mathbf{m}}(m) dm = E\{|\mathbf{m}|\}$, then

$$N_q = \frac{m_{\max}^2}{3L^2} \frac{\ln^2(1 + \mu)}{\mu^2} \left[1 + 2\mu \frac{E\{|\mathbf{m}|\}}{m_{\max}} + \mu^2 \frac{\sigma_{\mathbf{m}}^2}{m_{\max}^2} \right].$$

Differential Quantizers

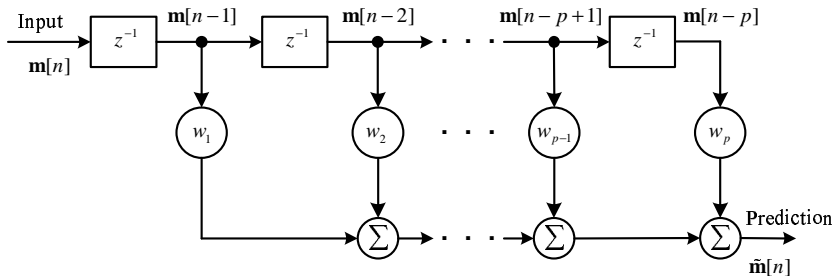
- Most message signals (e.g., voice or video) exhibit a high degree of correlation between successive samples.
- Redundancy can be exploited to obtain a better SNR_q for a given L , or conversely for a specified SNR_q the number of levels L can be reduced:
 - 1 Use the previous sample values to predict the next sample value and then transmit the difference.
 - 2 Quantize and transmit the prediction error,

$$\mathbf{e}[n] = \mathbf{m}[n] - \tilde{\mathbf{m}}[n].$$



If $|e_{\max}| = |m_{\max} - km_{\max}| = |1 - k|m_{\max}$ is less than m_{\max} then the quantization noise power is reduced!

Linear Predictor



Select $\{w_i\}$ to *minimize* the variance of prediction error:

$$\sigma_e^2 = E \left\{ \left(\mathbf{m}[n] - \sum_{i=1}^p w_i \mathbf{m}[n-i] \right)^2 \right\} = E\{\mathbf{m}^2[n]\} - 2 \sum_{i=1}^p w_i E\{\mathbf{m}[n]\mathbf{m}[n-i]\} + \sum_{i=1}^p \sum_{j=1}^p w_i w_j E\{\mathbf{m}[n-i]\mathbf{m}[n-j]\}.$$

