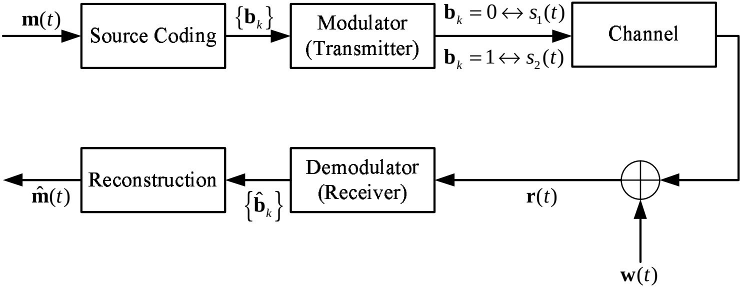
A First Course in Digital Communications

Ha H. Nguyen and E. Shwedyk

CAMBRIDGE UNIVERSITY PRESS

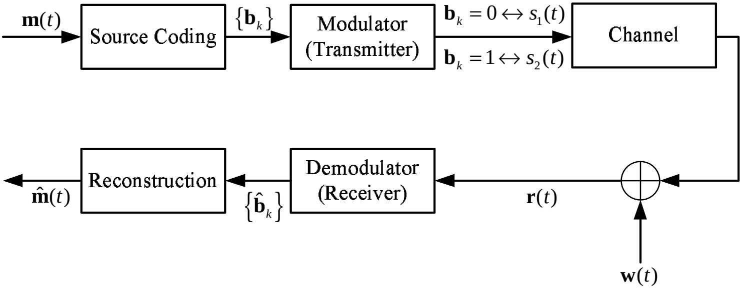
February 2009

Block Diagram of Binary Communication Systems



- Bits in two different time slots are *statistically independent*.
- *a priori* probabilities: $P[\mathbf{b}_k = 0] = P_1$, $P[\mathbf{b}_k = 1] = P_2$.
- Signals $s_1(t)$ and $s_2(t)$ have a duration of T_b seconds and finite energies: $E_1 = \int_0^{T_b} s_1^2(t)\mathrm{d}t$, $E_2 = \int_0^{T_b} s_2^2(t)\mathrm{d}t$.
- Noise $\mathbf{w}(t)$ is stationary *Gaussian*, zero-mean *white* noise with two-sided power spectral density of $N_0/2$ (watts/Hz):

$$E\{\mathbf{w}(t)\} = 0, \quad E\{\mathbf{w}(t)\mathbf{w}(t+\tau)\} = \frac{N_0}{2}\delta(\tau).$$



- Received signal over $[(k-1)T_b, kT_b]$:

$$\mathbf{r}(t) = s_i(t-(k-1)T_b) + \mathbf{w}(t), \quad (k-1)T_b \le t \le kT_b.$$

- Objective is to design a receiver (or demodulator) such that *the probability of making an error is minimized.*
- Shall reduce the problem from the observation of a time waveform to that of observing a set of numbers (which are random variables).

Geometric Representation of Signals $s_1(t)$ and $s_2(t)$ (I)

Wish to represent two arbitrary signals $s_1(t)$ and $s_2(t)$ as *linear combinations* of two *orthonormal* basis functions $\phi_1(t)$ and $\phi_2(t)$.

- $\phi_1(t)$ and $\phi_2(t)$ are orthonormal if:

$$\int_0^{T_b} \phi_1(t)\phi_2(t)\mathrm{d}t = 0 \text{ (\textit{ortho}gonality)},$$

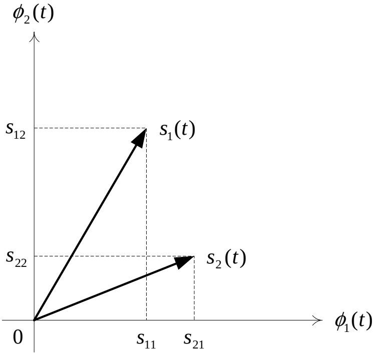
$$\int_0^{T_b} \phi_1^2(t)\mathrm{d}t = \int_0^{T_b} \phi_2^2(t)\mathrm{d}t = 1 \text{ (\textit{normal}ized to have unit energy)}.$$

- The representations are

$$\begin{aligned} s_1(t) &= s_{11}\phi_1(t) + s_{12}\phi_2(t), \\ s_2(t) &= s_{21}\phi_1(t) + s_{22}\phi_2(t). \end{aligned}$$

where $\quad s_{ij} = \int_0^{T_b} s_i(t)\phi_j(t)\mathrm{d}t, \quad i,j \in \{1,2\},$

Geometric Representation of Signals $s_1(t)$ and $s_2(t)$ (II)



$$s_1(t) = s_{11}\phi_1(t) + s_{12}\phi_2(t),$$
$$s_2(t) = s_{21}\phi_1(t) + s_{22}\phi_2(t),$$
$$s_{ij} = \int_0^{T_b} s_i(t)\phi_j(t)\mathrm{d}t,\ i, j \in \{1, 2\},$$

- $\int_0^{T_b} s_i(t)\phi_j(t)\mathrm{d}t$ is the projection of $s_i(t)$ onto $\phi_j(t)$.
- How to choose *orthonormal* functions $\phi_1(t)$ and $\phi_2(t)$ to represent $s_1(t)$ and $s_2(t)$ exactly?

Gram-Schmidt Procedure

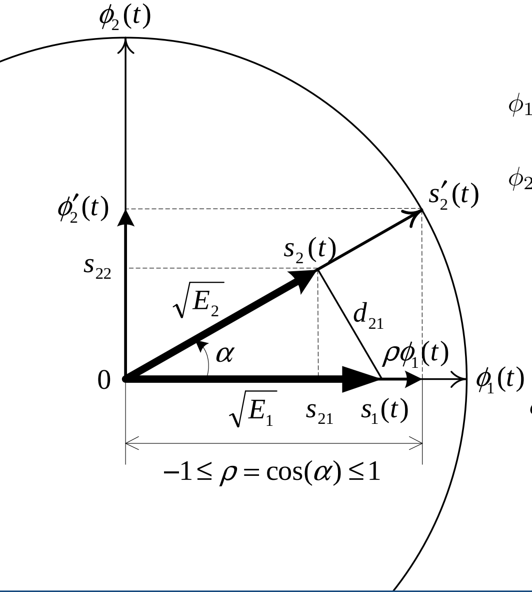
1. Let $\phi_1(t) \equiv \frac{s_1(t)}{\sqrt{E_1}}$. Note that $s_{11} = \sqrt{E_1}$ and $s_{12} = 0$.
2. Project $s_2'(t) = \frac{s_2(t)}{\sqrt{E_2}}$ onto $\phi_1(t)$ to obtain the *correlation coefficient*:

$$\rho = \int_0^{T_b} \frac{s_2(t)}{\sqrt{E_2}} \phi_1(t) \mathrm{d}t = \frac{1}{\sqrt{E_1 E_2}} \int_0^{T_b} s_1(t) s_2(t) \mathrm{d}t.$$

3. Subtract $\rho\phi_1(t)$ from $s_2'(t)$ to obtain $\phi_2'(t) = \frac{s_2(t)}{\sqrt{E_2}} - \rho\phi_1(t)$.
4. Finally, normalize $\phi_2'(t)$ to obtain:

$$\begin{aligned} \phi_2(t) &= \frac{\phi_2'(t)}{\sqrt{\int_0^{T_b} \left[\phi_2'(t)\right]^2 \mathrm{d}t}} = \frac{\phi_2'(t)}{\sqrt{1-\rho^2}} \\ &= \frac{1}{\sqrt{1-\rho^2}} \left[\frac{s_2(t)}{\sqrt{E_2}} - \frac{\rho s_1(t)}{\sqrt{E_1}} \right]. \end{aligned}$$

Gram-Schmidt Procedure: Summary



$$
\begin{aligned}
\phi_1(t) &= \frac{s_1(t)}{\sqrt{E_1}}, \\
\phi_2(t) &= \frac{1}{\sqrt{1-\rho^2}}\left[\frac{s_2(t)}{\sqrt{E_2}} - \frac{\rho s_1(t)}{\sqrt{E_1}}\right], \\
s_{21} &= \int_0^{T_b} s_2(t)\phi_1(t)\mathrm{d}t = \rho\sqrt{E_2}, \\
s_{22} &= \left(\sqrt{1-\rho^2}\right)\sqrt{E_2}, \\
d_{21} &= \sqrt{\int_0^{T_b}[s_2(t)-s_1(t)]^2\mathrm{d}t} \\
&= E_1 - 2\rho\sqrt{E_1E_2} + E_2.
\end{aligned}
$$

Gram-Schmidt Procedure for M Waveforms $\{s_i(t)\}_{i=1}^{M}$

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{\int_{-\infty}^{\infty} s_1^2(t)\mathrm{d}t}},$$

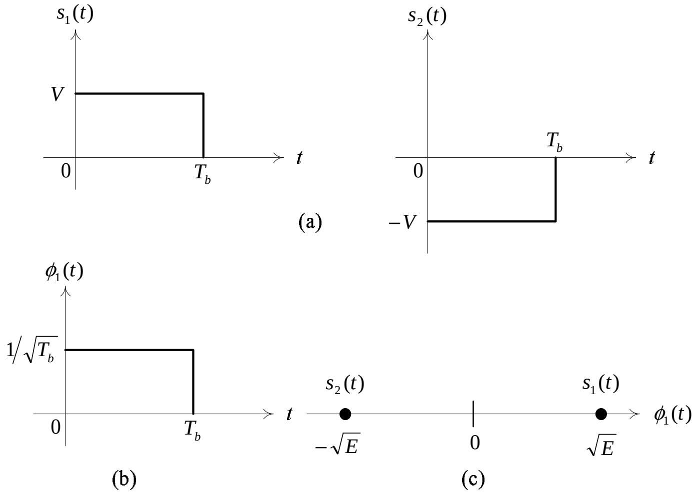
$$\phi_i(t) = \frac{\phi_i^{'}(t)}{\sqrt{\int_{-\infty}^{\infty}\left[\phi_i^{'}(t)\right]^2\mathrm{d}t}}, \quad i = 2, 3, \ldots, N,$$

$$\phi_i^{'}(t) = \frac{s_i(t)}{\sqrt{E_i}} - \sum_{j=1}^{i-1}\rho_{ij}\phi_j(t),$$

$$\rho_{ij} = \int_{-\infty}^{\infty}\frac{s_i(t)}{\sqrt{E_i}}\phi_j(t)\mathrm{d}t, \quad j = 1, 2, \ldots, i-1.$$

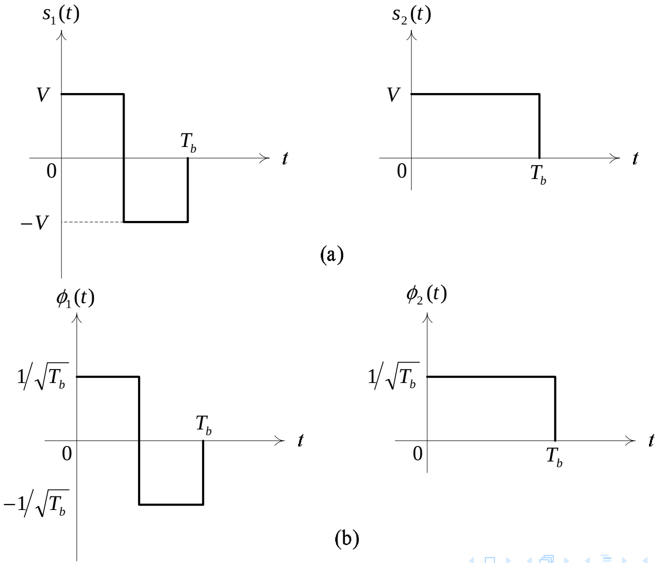
If the waveforms $\{s_i(t)\}_{i=1}^{M}$ form a *linearly independent set*, then $N = M$. Otherwise $N < M$.

Example 1



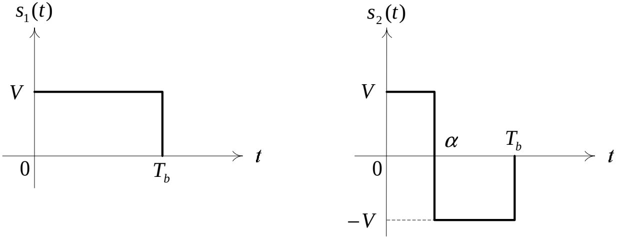
(a) Signal set. (b) Orthonormal function. (c) Signal space representation.

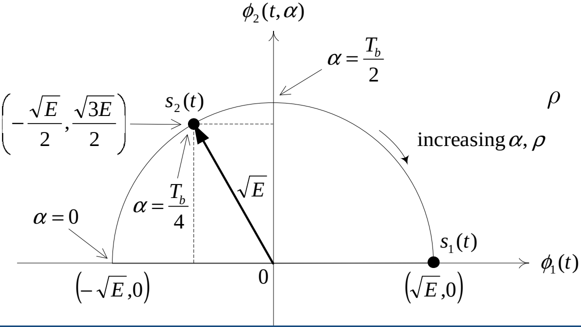
Example 2



$s_1(t)$
V
T_b
t
0
$-V$
$s_2(t)$
V
t
0
T_b
(a)
$\phi_1(t)$
$1/\sqrt{T_b}$
T_b
t
0
$-1/\sqrt{T_b}$
$\phi_2(t)$
$1/\sqrt{T_b}$
t
0
T_b
(b)

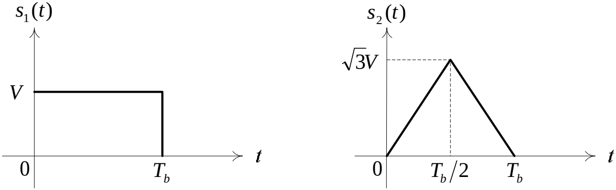
Example 3



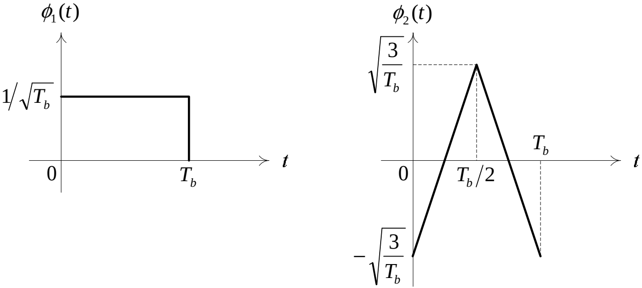


$$\begin{aligned}
\rho &= \frac{1}{E}\int_0^{T_b} s_2(t)s_1(t)\mathrm{d}t \\
&= \frac{1}{V^2T_b}\left[V^2\alpha - V^2(T_b-\alpha)\right] \\
&= \frac{2\alpha}{T_b} - 1
\end{aligned}$$

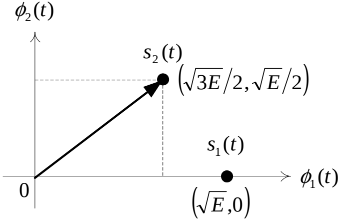
Example 4


$s_1(t)$
V
0
T_b
t
$s_2(t)$
$\sqrt{3}V$
0
$T_b/2$
T_b
t

(a)


$\phi_1(t)$
$1/\sqrt{T_b}$
0
T_b
t
$\phi_2(t)$
$\sqrt{\frac{3}{T_b}}$
T_b
0
$T_b/2$
t
$-\sqrt{\frac{3}{T_b}}$

(b)



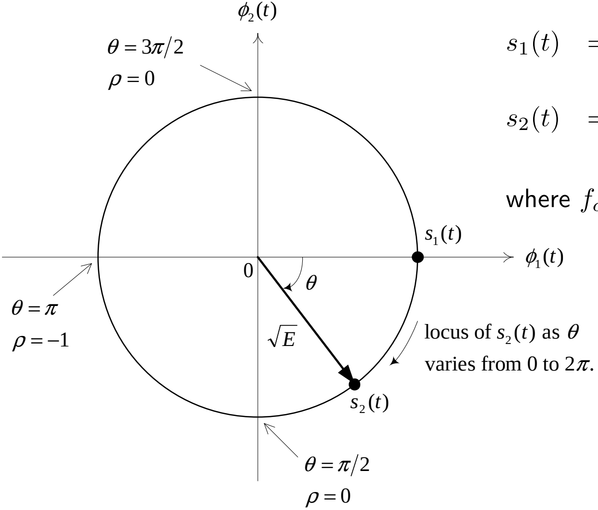
$$\rho = \frac{1}{E}\int_0^{T_b} s_2(t)s_1(t)\mathrm{d}t = \frac{2}{E}\int_0^{T_b/2}\left(\frac{2\sqrt{3}}{T_b}Vt\right)V\mathrm{d}t = \frac{\sqrt{3}}{2},$$

$$\phi_2(t) = \frac{1}{(1-\frac{3}{4})^{\frac{1}{2}}}\left[\frac{s_2(t)}{\sqrt{E}} - \rho\frac{s_1(t)}{\sqrt{E}}\right] = \frac{2}{\sqrt{E}}\left[s_2(t) - \frac{\sqrt{3}}{2}s_1(t)\right],$$

$$s_{21} = \frac{\sqrt{3}}{2}\sqrt{E}, \quad s_{22} = \frac{1}{2}\sqrt{E}.$$

$$d_{21} = \left[\int_0^{T_b}[s_2(t) - s_1(t)]^2\mathrm{d}t\right]^{\frac{1}{2}} = \sqrt{\left(2-\sqrt{3}\right)E}.$$

Example 5

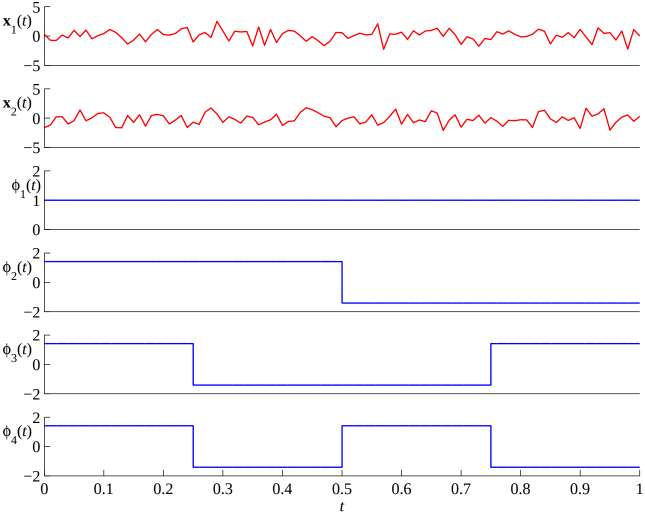


$$s_1(t) = \sqrt{E}\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t),$$

$$s_2(t) = \sqrt{E}\sqrt{\frac{2}{T_b}}\cos(2\pi f_c t + \theta).$$

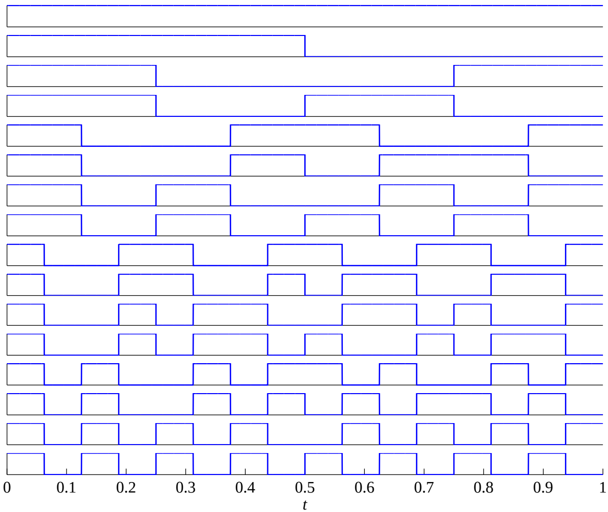
where $f_c = \frac{k}{2T_b}$, k an integer.

Representation of Noise with Walsh Functions



Exact representation of noise with 4 Walsh functions is not possible.

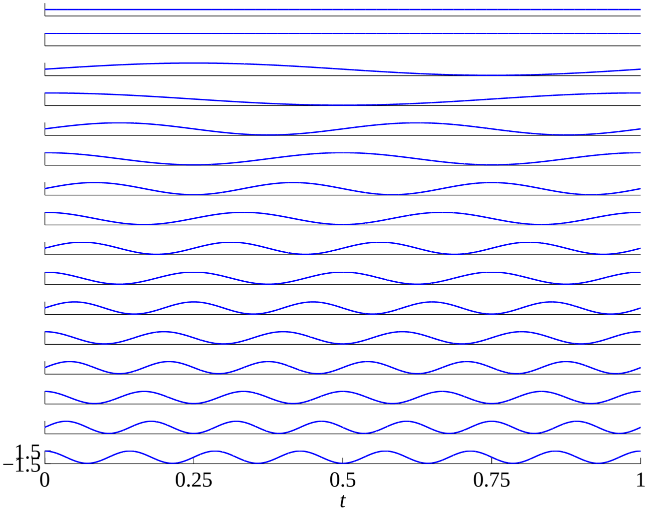
The First 16 Walsh Functions



Exact representations might be possible with many more Walsh functions.

The First 16 Sine and Cosine Functions

Can also use sine and cosine functions (Fourier representation).



Representation of the Noise I

- To represent the random noise signal, $\mathbf{w}(t)$, in the time interval $[(k-1)T_b, kT_b]$, need to use a *complete* orthonormal set of known deterministic functions:

$$\mathbf{w}(t) = \sum_{i=1}^{\infty} \mathbf{w}_i \phi_i(t), \quad \text{where} \quad \mathbf{w}_i = \int_0^{T_b} \mathbf{w}(t)\phi_i(t)\mathrm{d}t.$$

- The coefficients $\mathbf{w}_i$'s are *random variables* and understanding their statistical properties is imperative in developing the optimum receiver.

Representation of the Noise II

- When $\mathbf{w}(t)$ is zero-mean and white, then:
  1. $E\{\mathbf{w}_i\} = E\left\{\int_0^{T_b} \mathbf{w}(t)\phi_i(t)\mathrm{d}t\right\} = \int_0^{T_b} E\{\mathbf{w}(t)\}\phi_i(t)\mathrm{d}t = 0.$
  2. $E\{\mathbf{w}_i\mathbf{w}_j\} = E\left\{\int_0^{T_b} \mathrm{d}\lambda\mathbf{w}(\lambda)\phi_i(\lambda)\int_0^{T_b} \mathrm{d}\tau\mathbf{w}(\tau)\phi_j(\tau)\right\} = \begin{cases} \frac{N_0}{2}, & i = j \\ 0, & i \neq j \end{cases}.$

  $\{\mathbf{w}_1, \mathbf{w}_2, \ldots\}$ are zero-mean and uncorrelated random variables.
- If $\mathbf{w}(t)$ is not only zero-mean and white, but also Gaussian $\Rightarrow$ $\{\mathbf{w}_1, \mathbf{w}_2, \ldots\}$ are Gaussian and *statistically independent*!!!
- The above properties do not depend on how the set $\{\phi_i(t), i = 1, 2, \ldots\}$ is chosen.
- Shall choose as the first two functions the functions $\phi_1(t)$ and $\phi_2(t)$ used to represent the two signals $s_1(t)$ and $s_2(t)$ exactly. The remaining functions, i.e., $\phi_3(t)$, $\phi_4(t)$, ..., are simply chosen to complete the set.

Optimum Receiver I

Without any loss of generality, concentrate on the first bit interval. The received signal is

$$
\begin{aligned}
\mathbf{r}(t) &= s_i(t) + \mathbf{w}(t), \quad 0 \le t \le T_b \\
&= \begin{cases} s_1(t) + \mathbf{w}(t), & \text{if a ``0'' is transmitted} \\ s_2(t) + \mathbf{w}(t), & \text{if a ``1'' is transmitted} \end{cases} . \\
&= \underbrace{[s_{i1}\phi_1(t) + s_{i2}\phi_2(t)]}_{s_i(t)} \\
&\quad + \underbrace{[\mathbf{w}_1\phi_1(t) + \mathbf{w}_2\phi_2(t) + \mathbf{w}_3\phi_3(t) + \mathbf{w}_4\phi_4(t) + \cdots]}_{\mathbf{w}(t)} \\
&= (s_{i1} + \mathbf{w}_1)\phi_1(t) + (s_{i2} + \mathbf{w}_2)\phi_2(t) + \mathbf{w}_3\phi_3(t) + \mathbf{w}_4\phi_4(t) + \cdots \\
&= \mathbf{r}_1\phi_1(t) + \mathbf{r}_2\phi_2(t) + \mathbf{r}_3\phi_3(t) + \mathbf{r}_4\phi_4(t) + \cdots
\end{aligned}
$$

Optimum Receiver II

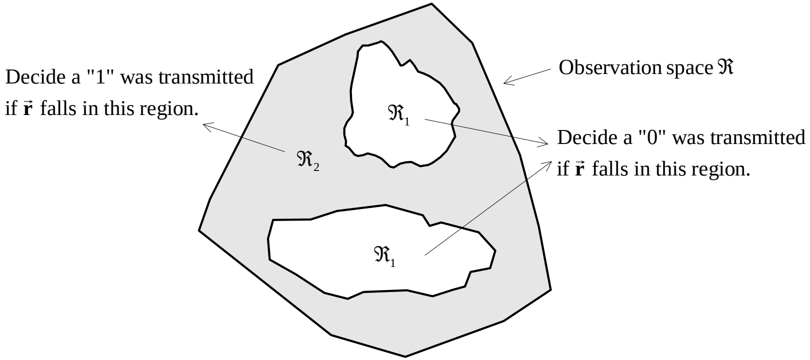
where $\mathbf{r}_j = \int_0^{T_b} \mathbf{r}(t)\phi_j(t)\mathrm{d}t$, and

$$\begin{aligned} \mathbf{r}_1 &= s_{i1} + \mathbf{w}_1 \\ \mathbf{r}_2 &= s_{i2} + \mathbf{w}_2 \\ \mathbf{r}_3 &= \mathbf{w}_3 \\ \mathbf{r}_4 &= \mathbf{w}_4 \\ &\vdots \end{aligned}$$

- Note that $\mathbf{r}_j$, for $j = 3, 4, 5, \ldots$, does not depend on which signal ($s_1(t)$ or $s_2(t)$) was transmitted.
- The decision can now be based on the observations $r_1, r_2, r_3, r_4, \ldots$.
- The criterion is *to minimize the bit error probability*.

Optimum Receiver III

- Consider only the first n terms (n can be very very large), $\vec{r} = \{r_1, r_2, \ldots, r_n\} \Rightarrow$ Need to partition the n-dimensional observation space into *decision regions*.



Optimum Receiver IV

$$
\begin{aligned}
P[\text{error}] = P[&(\text{“0” decided and “1” transmitted}) \text{ or} \\
&(\text{“1” decided and “0” transmitted})]. \\
= \; & P[0_D, 1_T] + P[1_D, 0_T] \\
= \; & P[0_D|1_T]P[1_T] + P[1_D|0_T]P[0_T] \\
= \; & P_2 \int_{\Re_1} f(\vec{r}|1_T)\mathrm{d}\vec{r} + P_1 \int_{\Re_2} f(\vec{r}|0_T)\mathrm{d}\vec{r} \\
= \; & P_2 \int_{\Re-\Re_2} f(\vec{r}|1_T)\mathrm{d}\vec{r} + P_1 \int_{\Re_2} f(\vec{r}|0_T)\mathrm{d}\vec{r} \\
= \; & P_2 \int_{\Re} f(\vec{r}|1_T)\mathrm{d}\vec{r} + \int_{\Re_2} [P_1 f(\vec{r}|0_T) - P_2 f(\vec{r}|1_T)]\mathrm{d}\vec{r} \\
= \; & P_2 + \int_{\Re_2} [P_1 f(\vec{r}|0_T) - P_2 f(\vec{r}|1_T)]\, \mathrm{d}\vec{r}.
\end{aligned}
$$

Optimum Receiver V

- The minimum error probability decision rule is

$$\begin{cases} P_1 f(\vec{r}|0_T) - P_2 f(\vec{r}|1_T) \geq 0 & \Rightarrow \text{ decide “0” } (0_D) \\ P_1 f(\vec{r}|0_T) - P_2 f(\vec{r}|1_T) < 0 & \Rightarrow \text{ decide “1” } (1_D) \end{cases}.$$

- Equivalently,

$$\frac{f(\vec{r}|1_T)}{f(\vec{r}|0_T)} \underset{0_D}{\overset{1_D}{\gtrless}} \frac{P_1}{P_2}. \tag{1}$$

- The expression $\frac{f(\vec{r}|1_T)}{f(\vec{r}|0_T)}$ is called the *likelihood ratio*.
- The decision rule in (1) was derived without specifying any statistical properties of the noise process $\mathbf{w}(t)$.

Optimum Receiver VI

- Simplified decision rule when the noise $\mathbf{w}(t)$ is zero-mean, white and Gaussian:

$$(r_1 - s_{11})^2 + (r_2 - s_{12})^2 \underset{0_D}{\overset{1_D}{\gtrless}} (r_1 - s_{21})^2 + (r_2 - s_{22})^2 + N_0 \ln\left(\frac{P_1}{P_2}\right).$$

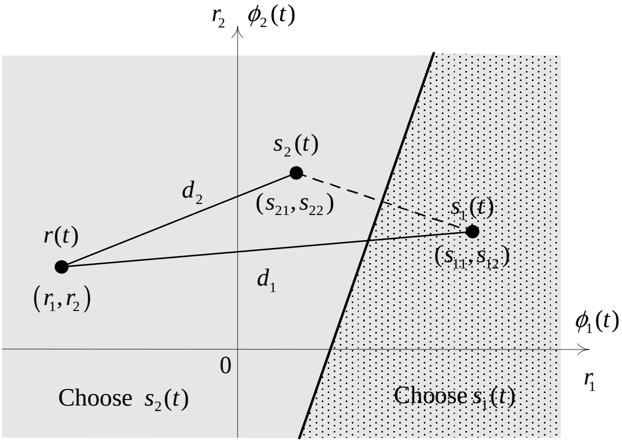
- For the special case of $P_1 = P_2$ (signals are equally likely):

$$(r_1 - s_{11})^2 + (r_2 - s_{12})^2 \underset{0_D}{\overset{1_D}{\gtrless}} (r_1 - s_{21})^2 + (r_2 - s_{22})^2.$$

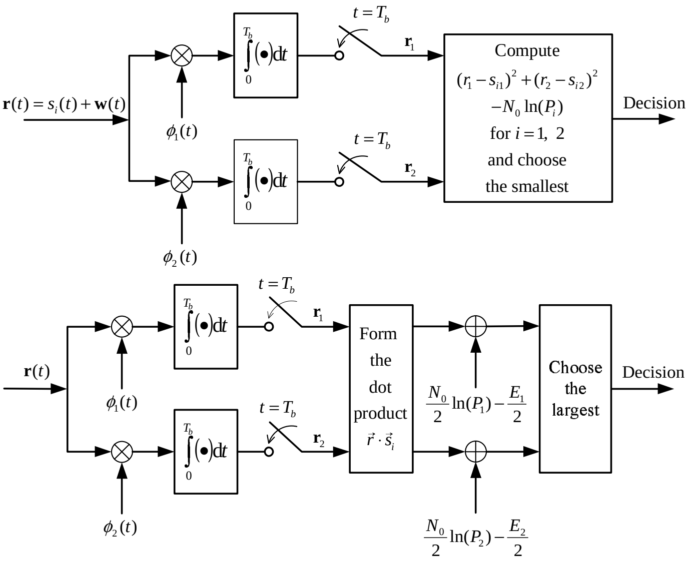
$\Rightarrow$ *minimum-distance* receiver!

Minimum-Distance Receiver

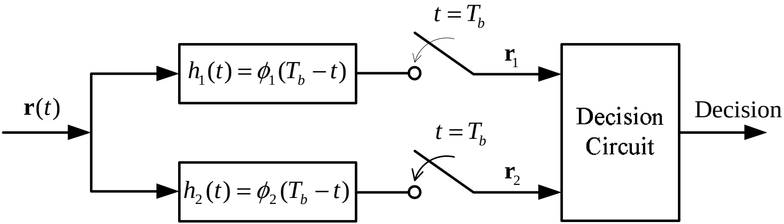
$$(r_1 - s_{11})^2 + (r_2 - s_{12})^2 \underset{0_D}{\overset{1_D}{\gtrless}} (r_1 - s_{21})^2 + (r_2 - s_{22})^2.$$



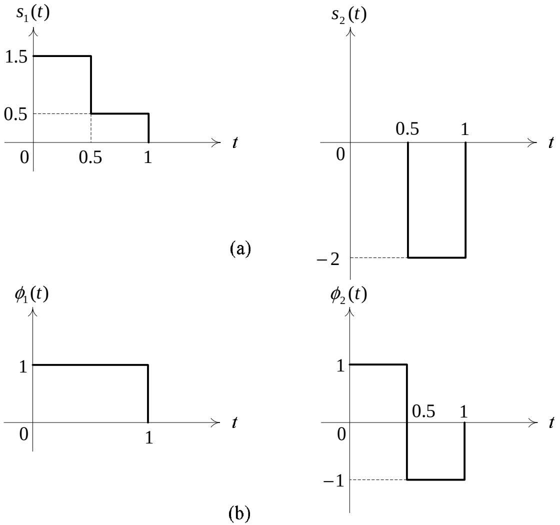
Correlation Receiver Implementation



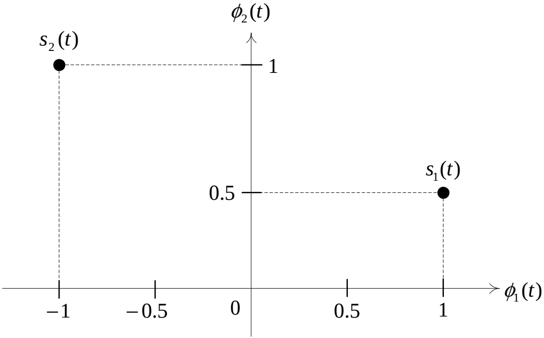
Receiver Implementation using Matched Filters



Example 5.6 I

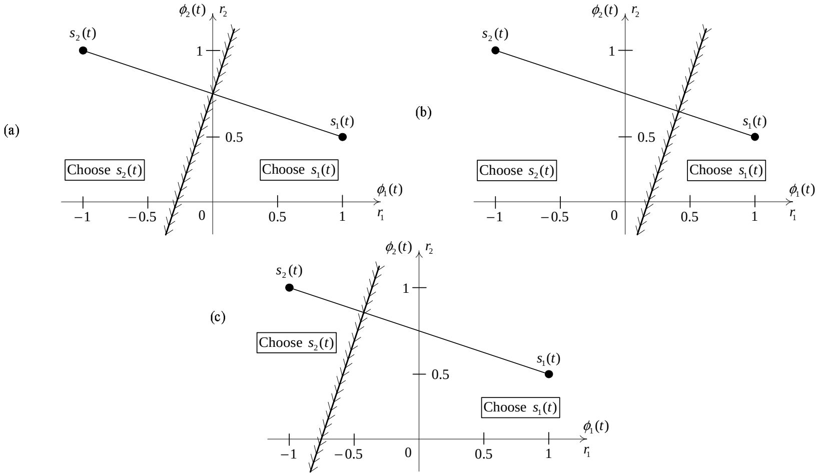


Example 5.6 II



$$
\begin{aligned}
s_1(t) &= \phi_1(t) + \frac{1}{2}\phi_2(t), \\
s_2(t) &= -\phi_1(t) + \phi_2(t).
\end{aligned}
$$

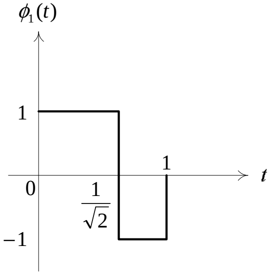
Example 5.6 III

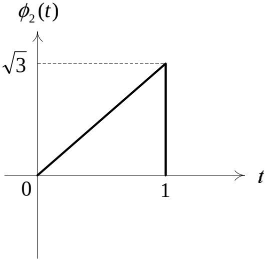


(a) $P_1 = P_2 = 0.5$, (b) $P_1 = 0.25,\ P_2 = 0.75$. (c) $P_1 = 0.75,\ P_2 = 0.25$.

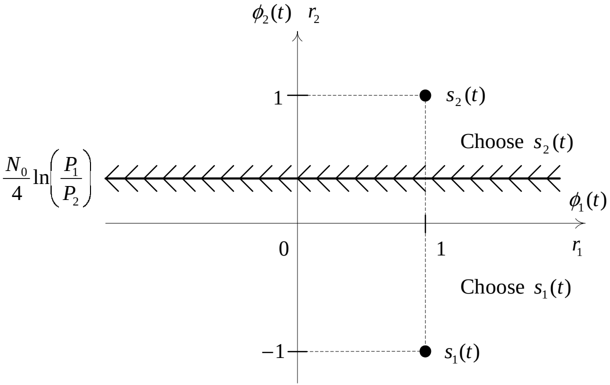
Example 5.7 I

$$s_2(t) = \phi_1(t) + \phi_2(t),$$
$$s_1(t) = \phi_1(t) - \phi_2(t).$$

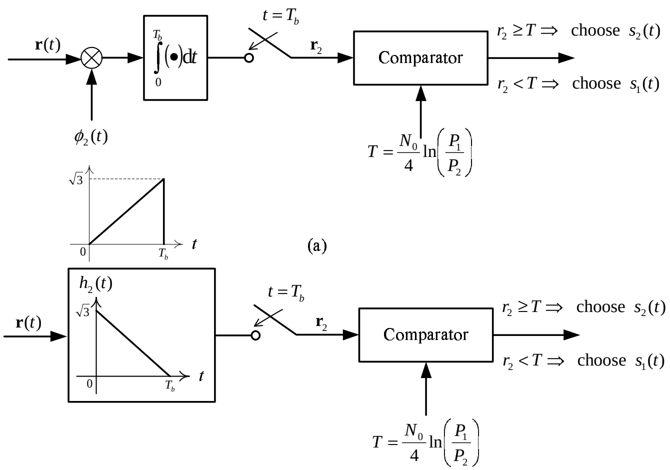




Example 5.7 II



Example 5.7 III

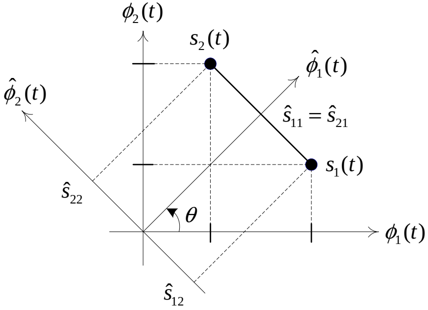


(a)

(b)

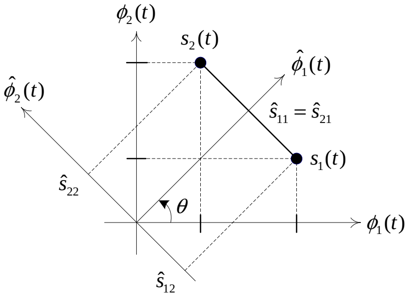
Implementation with One Correlator/Matched Filter

Always possible by a judicious choice of the orthonormal basis.



(b)

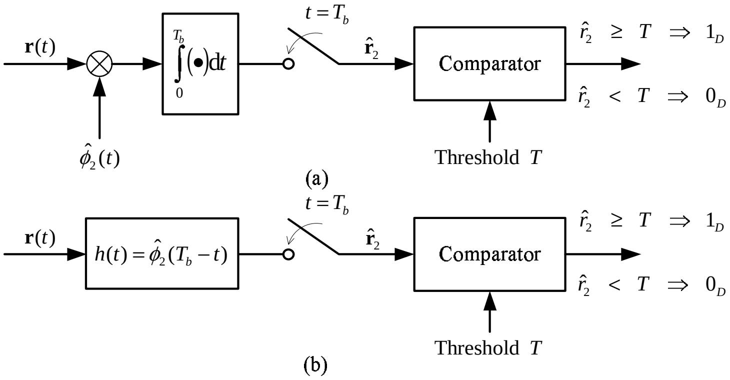
$$\begin{bmatrix} \hat{\phi}_1(t) \\ \hat{\phi}_2(t) \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix}.$$



(b)

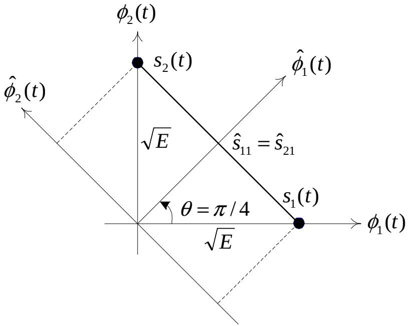
$$\frac{f(\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots, |1_T)}{f(\hat{r}_1, \hat{r}_2, \hat{r}_3, \ldots, |0_T)} = \frac{f(\hat{s}_{21} + \hat{w}_1) f(\hat{s}_{22} + \hat{w}_2) f(\hat{w}_3) \ldots}{f(\hat{s}_{11} + \hat{w}_1) f(\hat{s}_{12} + \hat{w}_2) f(\hat{w}_3) \ldots} \quad \overset{1_D}{\underset{0_D}{\gtrless}} \quad \frac{P_1}{P_2}$$

$$\hat{r}_2 \quad \overset{1_D}{\underset{0_D}{\gtrless}} \quad \frac{\hat{s}_{22} + \hat{s}_{12}}{2} + \left(\frac{N_0/2}{\hat{s}_{22} - \hat{s}_{12}} \right) \ln \left(\frac{P_1}{P_2} \right) \equiv T.$$



$$\hat{\phi}_2(t) = \frac{s_2(t) - s_1(t)}{\left(E_2 - 2\rho\sqrt{E_1 E_2} + E_1\right)^{\frac{1}{2}}}, \; T \equiv \frac{\hat{s}_{22} + \hat{s}_{12}}{2} + \left(\frac{N_0/2}{\hat{s}_{22} - \hat{s}_{12}}\right) \ln\left(\frac{P_1}{P_2}\right).$$

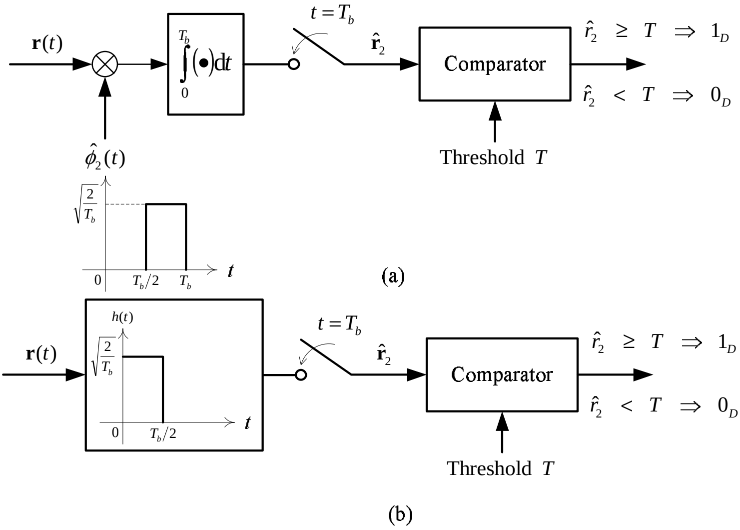
Example 5.8 I



$$\hat{\phi}_1(t) = \frac{1}{\sqrt{2}}[\phi_1(t) + \phi_2(t)],$$

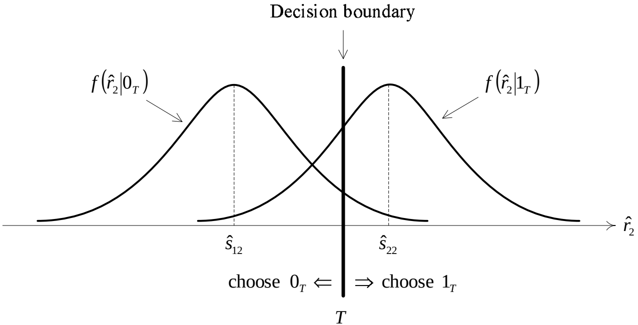
$$\hat{\phi}_2(t) = \frac{1}{\sqrt{2}}[-\phi_1(t) + \phi_2(t)].$$

Example 5.8 II

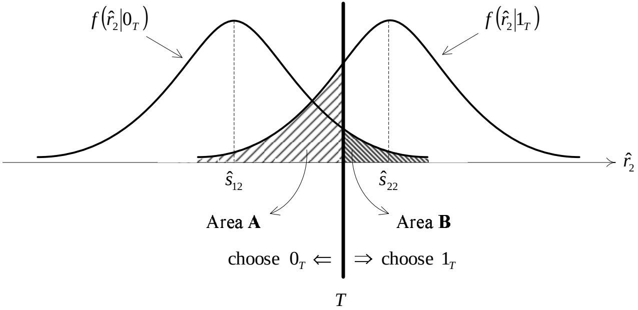


Receiver Performance

To detect $\mathbf{b}_k$, compare $\hat{\mathbf{r}}_2 = \int_{(k-1)T_b}^{kT_b} \mathbf{r}(t)\hat{\phi}_2(t)\mathrm{d}t$ to the threshold $T = \frac{\hat{s}_{12}+\hat{s}_{22}}{2} + \frac{N_0}{2(\hat{s}_{22}-\hat{s}_{12})}\ln\left(\frac{P_1}{P_2}\right)$.

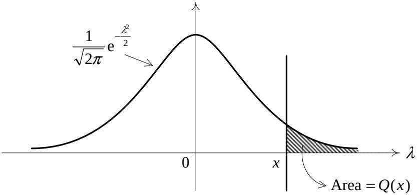


$$
\begin{aligned}
P[\text{error}] &= P[(\text{0 transmitted and 1 decided) or (1 transmitted and 0 decided})] \\
&= P[(0_T, 1_D) \text{ or } (1_T, 0_D)].
\end{aligned}
$$

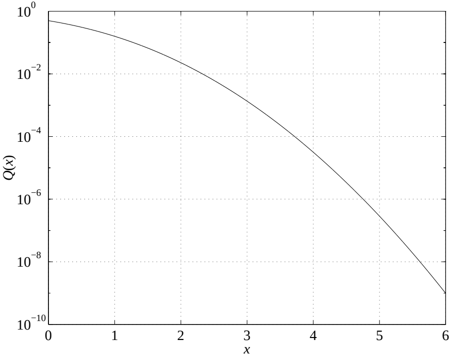


$$
\begin{aligned}
P[\text{error}] &= P[0_T, 1_D] + P[1_T, 0_D] = P[1_D|0_T]P[0_T] + P[0_D|1_T]P[1_T] \\
&= P_1 \underbrace{\int_T^{\infty} f(\hat{r}_2|0_T)\mathrm{d}\hat{r}_2}_{\text{Area } \mathbf{B}} + P_2 \underbrace{\int_{-\infty}^{T} f(\hat{r}_2|1_T)\mathrm{d}\hat{r}_2}_{\text{Area } \mathbf{A}} \\
&= P_1 Q\left(\frac{T - \hat{s}_{12}}{\sqrt{N_0/2}}\right) + P_2\left[1 - Q\left(\frac{T - \hat{s}_{22}}{\sqrt{N_0/2}}\right)\right].
\end{aligned}
$$

Q-function



$$Q(x) \equiv \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp\left(-\frac{\lambda^2}{2}\right) \mathrm{d}\lambda.$$



Performance when $P_1 = P_2$

$$P[\text{error}] = Q\left(\frac{\hat{s}_{22} - \hat{s}_{12}}{2\sqrt{N_0/2}}\right) = Q\left(\frac{\text{distance between the signals}}{2 \times \text{noise RMS value}}\right).$$

- Probability of error decreases as either the two signals become more dissimilar (increasing the distances between them) or the noise power becomes less.
- To maximize the distance between the two signals one chooses them so that they are placed 180° from each other $\Rightarrow$ $s_2(t) = -s_1(t)$, i.e., *antipodal signaling*.
- The error probability does *not* depend on the signal shapes but only on the distance between them.

Relationship Between $Q(x)$ and $\text{erfc}(x)$.

- The *complementary error function* is defined as:

$$\begin{aligned} \text{erfc}(x) &= \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} \exp(-\lambda^2) \mathrm{d}\lambda \\ &= 1 - \text{erf}(x). \end{aligned}$$

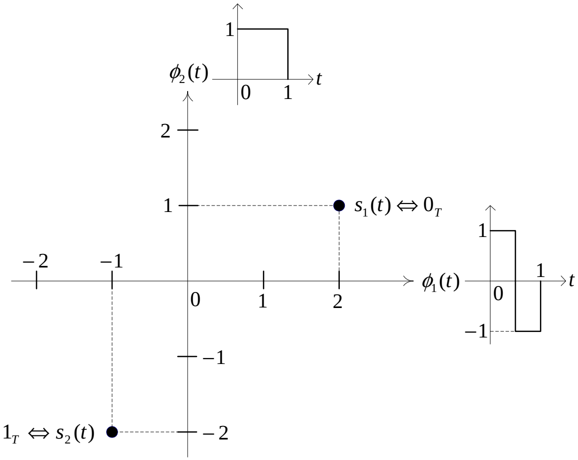
- erfc-function and the Q-function are related by:

$$\begin{aligned} Q(x) &= \frac{1}{2} \text{erfc}\left(\frac{x}{\sqrt{2}}\right) \\ \text{erfc}(x) &= 2Q(\sqrt{2}x). \end{aligned}$$

- Let $Q^{-1}(x)$ and $\text{erfc}^{-1}(x)$ be the inverses of $Q(x)$ and $\text{erfc}(x)$, respectively. Then

$$Q^{-1}(x) = \sqrt{2}\,\text{erfc}^{-1}(2x).$$

Example 5.9 I



$\phi_2(t)$
1
0
1
t
2
1
$s_1(t) \Leftrightarrow 0_T$
−2
−1
0
1
2
$\phi_1(t)$
1
1
t
0
−1
−1
$1_T \Leftrightarrow s_2(t)$
−2

Example 5.9 II

(a) Determine and sketch the two signals $s_1(t)$ and $s_2(t)$.

(b) The two signals $s_1(t)$ and $s_2(t)$ are used for the transmission of equally likely bits 0 and 1, respectively, over an additive white Gaussian noise (AWGN) channel. Clearly draw the decision boundary and the decision regions of the optimum receiver. Write the expression for the optimum decision rule.

(c) Find and sketch the two orthonormal basis functions $\hat{\phi}_1(t)$ and $\hat{\phi}_2(t)$ such that the optimum receiver can be implemented using only the projection $\hat{\mathbf{r}}_2$ of the received signal $\mathbf{r}(t)$ onto the basis function $\hat{\phi}_2(t)$. Draw the block diagram of such a receiver that uses a matched filter.

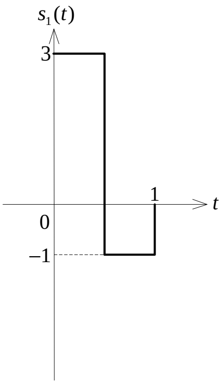
Example 5.9 III

(d) Consider now the following argument put forth by your classmate. She reasons that since the component of the signals along $\hat{\phi}_1(t)$ is not useful at the receiver in determining which bit was transmitted, one should not even transmit this component of the signal. Thus she modifies the transmitted signal as follows:

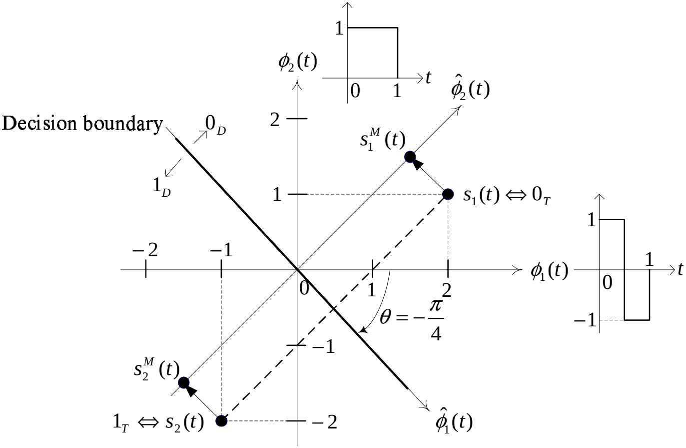
$$\begin{aligned} s_1^{(\mathsf{M})}(t) &= s_1(t) - \Big(\text{component of } s_1(t) \text{ along } \hat{\phi}_1(t)\Big) \\ s_2^{(\mathsf{M})}(t) &= s_2(t) - \Big(\text{component of } s_2(t) \text{ along } \hat{\phi}_1(t)\Big) \end{aligned}$$

Clearly identify the locations of $s_1^{(\mathsf{M})}(t)$ and $s_2^{(\mathsf{M})}(t)$ in the signal space diagram. What is the average energy of this signal set? Compare it to the average energy of the original set. Comment.

Example 5.9 IV



Example 5.9 V

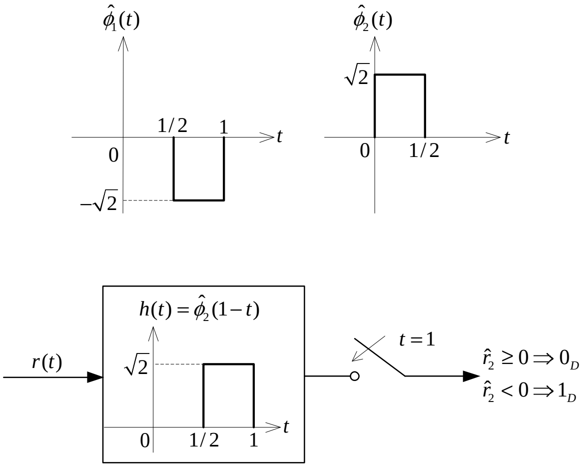

Decision boundary
0_D
1_D
$\phi_2(t)$
$\hat{\phi}_2(t)$
$s_1^M(t)$
$s_1(t) \Leftrightarrow 0_T$
$\phi_1(t)$
$\theta = -\frac{\pi}{4}$
$s_2^M(t)$
$1_T \Leftrightarrow s_2(t)$
$\hat{\phi}_1(t)$
t
1
0
−1
2
−2

Example 5.9 VI

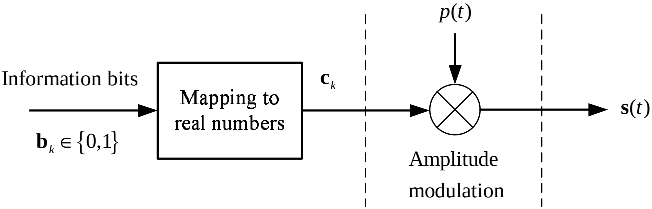
$$\begin{bmatrix} \hat{\phi}_1(t) \\ \hat{\phi}_2(t) \end{bmatrix} = \begin{bmatrix} \cos(-\pi/4) & \sin(-\pi/4) \\ -\sin(-\pi/4) & \cos(-\pi/4) \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \phi_1(t) \\ \phi_2(t) \end{bmatrix}.$$

$$\hat{\phi}_1(t) = \frac{1}{\sqrt{2}}[\phi_1(t) - \phi_2(t)],$$
$$\hat{\phi}_2(t) = \frac{1}{\sqrt{2}}[\phi_1(t) + \phi_2(t)].$$

Example 5.9 VII



PSD of Digital Amplitude Modulation I



- $\mathbf{c}_k$ is drawn from a finite set of real numbers with a probability that is known.
- Examples: $\mathbf{c}_k \in \{-1, +1\}$ (antipodal signaling), $\{0, 1\}$ (on-off keying), $\{-1, 0, +1\}$ (pseudoternary line coding) or $\{\pm 1, \pm 3, \cdots, \pm(M-1)\}$ (M-ary amplitude-shift keying).
- $p(t)$ is a pulse waveform of duration T_b.

PSD of Digital Amplitude Modulation II

- The transmitted signal is

$$\mathbf{s}(t) = \sum_{k=-\infty}^{\infty} \mathbf{c}_k p(t - kT_b).$$

- To find PSD, truncate the random process to a time interval of $-T = -NT_b$ to $T = NT_b$:

$$\mathbf{s}_T(t) = \sum_{k=-N}^{N} \mathbf{c}_k p(t - kT_b).$$

- Take the Fourier transform of the truncated process:

$$\mathbf{S}_T(f) = \sum_{k=-\infty}^{\infty} \mathbf{c}_k \mathcal{F}\{p(t - kT_b)\} = P(f) \sum_{k=-\infty}^{\infty} \mathbf{c}_k \mathrm{e}^{-j2\pi f k T_b}.$$

PSD of Digital Amplitude Modulation III

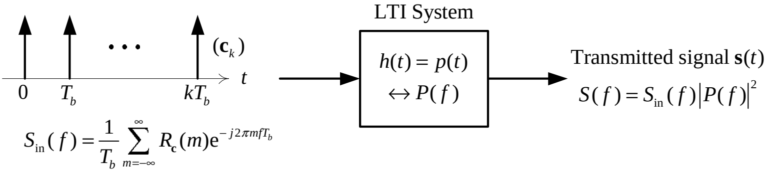
- Apply the basic definition of PSD:

$$
\begin{aligned}
S(f) &= \lim_{T\to\infty} \frac{E\left\{\left|\mathbf{S}_T(f)\right|^2\right\}}{2T} \\
&= \lim_{N\to\infty} \frac{|P(f)|^2}{(2N+1)T_b} E\left\{\left|\sum_{k=-N}^{N} \mathbf{c}_k \mathrm{e}^{-j2\pi f k T_b}\right|^2\right\}. \\
&= \frac{|P(f)|^2}{T_b} \sum_{m=-\infty}^{\infty} R_{\mathbf{c}}(m) \mathrm{e}^{-j2\pi m f T_b}.
\end{aligned}
$$

where $R_{\mathbf{c}}(m) = E\{\mathbf{c}_k \mathbf{c}_{k-m}\}$ is the (discrete) autocorrelation of $\{\mathbf{c}_k\}$, with $R_{\mathbf{c}}(m) = R_{\mathbf{c}}(-m)$.

PSD of Digital Amplitude Modulation IV

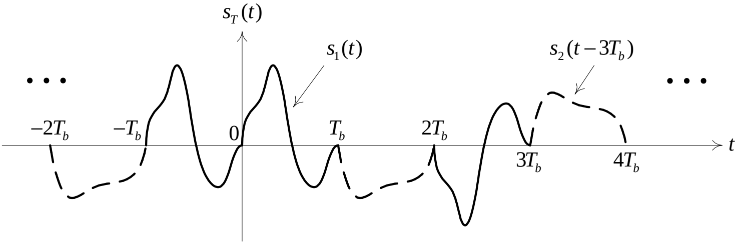
- The output PSD is the input PSD multiplied by $|P(f)|^2$, a transfer function.



$$S(f) = \frac{|P(f)|^2}{T_b} \sum_{m=-\infty}^{\infty} R_{\mathbf{c}}(m)\mathrm{e}^{-j2\pi mfT_b}.$$

PSD Derivation of Arbitrary Binary Modulation I

- Applicable to *any* binary modulation with *arbitrary* a priori probabilities, but restricted to *statistically independent* bits.



$$\mathbf{s}_T(t)=\sum_{k=-\infty}^{\infty}\mathbf{g}_k(t),\ \mathbf{g}_k(t)=\begin{cases} s_1(t-kT_b), & \text{with probability } P_1 \\ s_2(t-kT_b), & \text{with probability } P_2 \end{cases}.$$

PSD Derivation of Arbitrary Binary Modulation II

Decompose $\mathbf{s}_T(t)$ into a sum of a DC and an AC component:

$$\mathbf{s}_T(t) = \underbrace{E\{\mathbf{s}_T(t)\}}_{\text{DC}} + \underbrace{\mathbf{s}_T(t) - E\{\mathbf{s}_T(t)\}}_{\text{AC}} = v(t) + \mathbf{q}(t)$$

$$v(t) = E\{\mathbf{s}_T(t)\} = \sum_{k=-\infty}^{\infty} [P_1 s_1(t - kT_b) + P_2 s_2(t - kT_b)]$$

$$S_v(f) = \sum_{n=-\infty}^{\infty} |D_n|^2 \delta\left(f - \frac{n}{T_b}\right), \; D_n = \frac{1}{T_b}\left[P_1 S_1\left(\frac{n}{T_b}\right) + P_2 S_2\left(\frac{n}{T_b}\right)\right]$$

where $S_1(f)$ and $S_2(f)$ are the FTs of $s_1(t)$ and $s_2(t)$.

$$S_v(f) = \sum_{n=-\infty}^{\infty} \left| \frac{P_1 S_1\left(\frac{n}{T_b}\right) + P_2 S_2\left(\frac{n}{T_b}\right)}{T_b} \right|^2 \delta\left(f - \frac{n}{T_b}\right).$$

PSD Derivation of Arbitrary Binary Modulation III

To calculate $S_{\mathbf{q}}(f)$, apply the basic definition of PSD:

$$S_{\mathbf{q}}(f) = \lim_{T\to\infty} \frac{E\{|\mathbf{G}_T(f)|^2\}}{T} = \cdots = \frac{P_1 P_2}{T_b}|S_1(f) - S_2(f)|^2.$$

Finally,

$$\begin{aligned} S_{\mathbf{s}_T}(f) &= \frac{P_1 P_2}{T_b}|S_1(f) - S_2(f)|^2 + \\ & \sum_{n=-\infty}^{\infty} \left| \frac{P_1 S_1\left(\frac{n}{T_b}\right) + P_2 S_2\left(\frac{n}{T_b}\right)}{T_b} \right|^2 \delta\left(f - \frac{n}{T_b}\right). \end{aligned}$$